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AUTHOR Beck, A.; And Others

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ABSTRACT

This is part two of a three-part SMSG calculus text for high school students. One of the goals of the text is to present calculus as a mathematical discipline as well as presenting its practical uses. The authors emphasize the importance of being able to interpret the concepts and theory in terms of models to which they apply. The text demonstrates the origins of the ideas of the calculus in practical problems; attempts to express these ideas precisely and develop them logically; and finally, returns to the problems and applies the theorems resulting from that development. Chapter topics include: (1) area and integral; (2) basic integral theorems; (3) logarithmic and exponential functions; (4) growth, decay, and competition; and (5) integration. (MP)

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Calculus

Part 2 Student's Text

REVISED EDITION

The following is a list of all those who participated in the preparation of this volume:

A. Beck	Olney High School, Philadelphia, Pa.
A. A. Blank	New York University, New York, N.Y.
F. L. Elder	West Hempstead Jr., Sr. High School, N.Y.
C. E. Kerr	Dickinson College, Carlisle, Pa.
M. S. Klamkin	Ford Scientific Laboratory, Dearborn, Mich.
I. I. Kolodner	Carnegie Institute of Technology, Pittsburgh, Pa.
M. D. Kruskal	Princeton University, Princeton, N.J.
C. W. Leeds, III	Berkshire School, Sheffield, Mass.
M. A. Linton, Jr.	William Penn Charter School, Philadelphia, Pa.
H. M. Marston	Douglass College, New Brunswick, N.J.
I. Marx	Purdue University, Lafayette, Ind.
R. Pollack	New York University, New York, N.Y.
T. L. Reynolds	College of William and Mary, Williamsburg, Va.
R. L. Starkey	Cubberley High School, Palo Alto, Calif.
V. Twersky	Sylvania Electronics Defense Labs., Mt. View, Calif.
H. Weitzner	New York University, New York, N.Y.

Stanford, California
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Chapter 6

AREA AND INTEGRAL

6-1. Introduction.

Area, as we treated the idea in Section 1-2, was not defined analytically but accepted as a geometrically understood concept. We did not question the idea that a region with a curved boundary has a definite area but began with the implicit assumption that it does. Within the framework of our elementary knowledge we saw no way to describe the area of such a region except as a limit. For this purpose we used a specific kind of limit, the integral. Having gone from the geometrical description of area to an analytical method which determines its numerical value we are now able to use the analytical method to define the concept of area. In this chapter we shall take the concept of area arrived at intuitively and express it in precise analytical terms.

Underlying our method for determining the area of a region, there are a few elementary ideas. These ideas are commonly accepted properties of area which we postulate as the basis for the formal analytical definition of area. The area function α which associates with each region R of the plane a real number, the area of R, should satisfy the following properties.

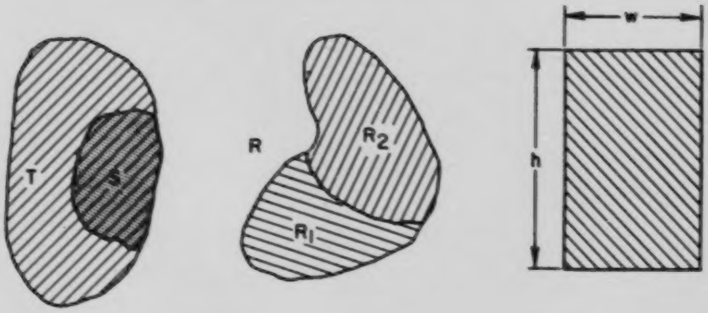
Property 1. $\alpha(R) \geq 0$

Property 2. If S and T are two regions and if S is contained in T (every point of S is also a point of T) then $\alpha(S) \leq \alpha(T)$.

Property 3. If R is the union of two nonoverlapping regions R_1 and R_2 (every point of R lies in R_1 or R_2 and only the points on their common boundary lie in both R_1 and R_2), then $\alpha(R) = \alpha(R_1) + \alpha(R_2)$.

Property 4. If R is a rectangle of height h and width w then $\alpha(R) = hw$.

Property 2 is called the order property of area and Property 3 the additive property. Properties 2-4 are illustrated in Figure 6-1a.



Property 2 Property 3 Property 4

Figure 6-1a

We do not expect to be able to define an area for every set of points in the plane. Consider the sets

$$S = \{(x,y) : 0 \le x \le 1 \, , \, 0 \le y \le g(x)\}$$
$$T = \{(x,y) : 0 \le x \le 1 \, , \, 0 \le y \le f(x)\}$$

where g is the Weierstrass function described in Section A4-3, and f is the function given by

$$f(x) = \begin{cases} 0 \, , \text{ for } x \text{ rational} \\ 1 \, , \text{ for } x \text{ irrational} \end{cases}$$

(Exercises A6-2, No. 4). For the present, it is far from clear that an area can be assigned to either of these regions in a meaningful way.

Exercises 6-1

1. Read Section 1-2 carefully and locate the places in the discussion where the four properties of area are used.

2. Prove from Property 3 that if a region R is the union of n nonoverlapping regions then

$$\alpha(R) = \alpha(R_1) + \alpha(R_2) + \dots + \alpha(R_n) \; .$$

3. Show that Property 2 is actually a consequence of Property 3 given that area is nonnegative.

4. (a) Using the given properties of area obtain the area of a triangle by elementary geometrical arguments.
   (b) Do the same for a trapezoid.

5. Estimate the area of each region described below.

   (a) $\{(x,y) : y \geq 0\} \cap \{(x,y) : y \leq 1 - x^2\}$

   (b) $\{(x,y) : 0 \leq y \leq \frac{1}{1 + x^2}\} \cap \{(x,y) : 0 \leq x \leq 1\}$

6. If Property 4 is replace by

   Property 4^*. The area of a unit square is one,

   Property 5. Congruent regions have the same area,

   show that the area of a square whose side is of length a is a^2.

7. Using Number 6, show that the area of a rectangle of height h and width w is hw.

6-2. Evaluation of an Area.

In Section 1-2 we reduced the problem of calculating the area bounded by a curve to the problem of determining the areas of certain standard regions. Let f be a nonnegative bounded function defined on $[a,b]$. We recall that the standard region R under the graph of f on $[a,b]$ is the set of points bounded above by the graph of f, below by the x-axis, on the left by the vertical line $x = a$ and on the right by $x = b$; that is,

$$R = \{(x,y) : a \leq x \leq b \text{ and } 0 \leq y \leq f(x)\}$$

(Figure 6-2a). To estimate the area of R we subdivided the standard region into smaller standard regions by subdividing the base interval $[a,b]$.

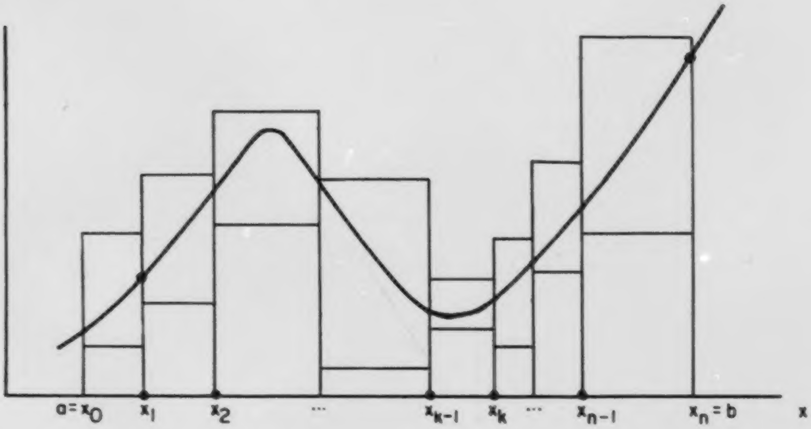


Figure 6-2a

We subdivide the interval into n parts, setting $x_0 = a$, $x_n = b$ and choosing points of subdivision $x_1, x_2, \ldots, x_{n-1}$ such that

$$x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n .$$

On each interval $[x_{k-1}, x_k]$, where $k = 1, 2, \ldots, n$, we have a standard region R_k where

$$R_k = \{(x,y) : x_{k-1} \leq x \leq x_k \text{ and } 0 \leq y \leq f(x)\} .$$

We then estimate the area of each subregion R_k from above and below by rectangular approximations. In each interval $[x_{k-1}, x_k]$ we obtain a lower bound m_k and an upper bound M_k for $f(x)$:

$$m_k \leq f(x) \leq M_k , \qquad (x_{k-1} \leq x \leq x_k) .$$

The region R_k is therefore contained in a rectangle of height M_k and, in turn, contains a rectangle of height m_k on the common base $[x_{k-1}, x_k]$. We conclude from Property 2 and Property 4 (Section 6-1), that

$$m_k(x_k - x_{k-1}) \leq \alpha(R_k) \leq M_k(x_k - x_{k-1}) .$$

Using the additive property, Property 3, we then have

$$\alpha(R) = \alpha(R_1) + \alpha(R_2) + \dots + \alpha(R_n) .$$

It follows that

$$\alpha(R) \geq m_1(x_1 - x_0) + m_2(x_2 - x_1) + \dots + m_n(x_n - x_{n-1})$$

and

$$\alpha(R) \leq M_1(x_1 - x_0) + M_2(x_2 - x_1) + \dots + M_n(x_n - x_{n-1}) .$$

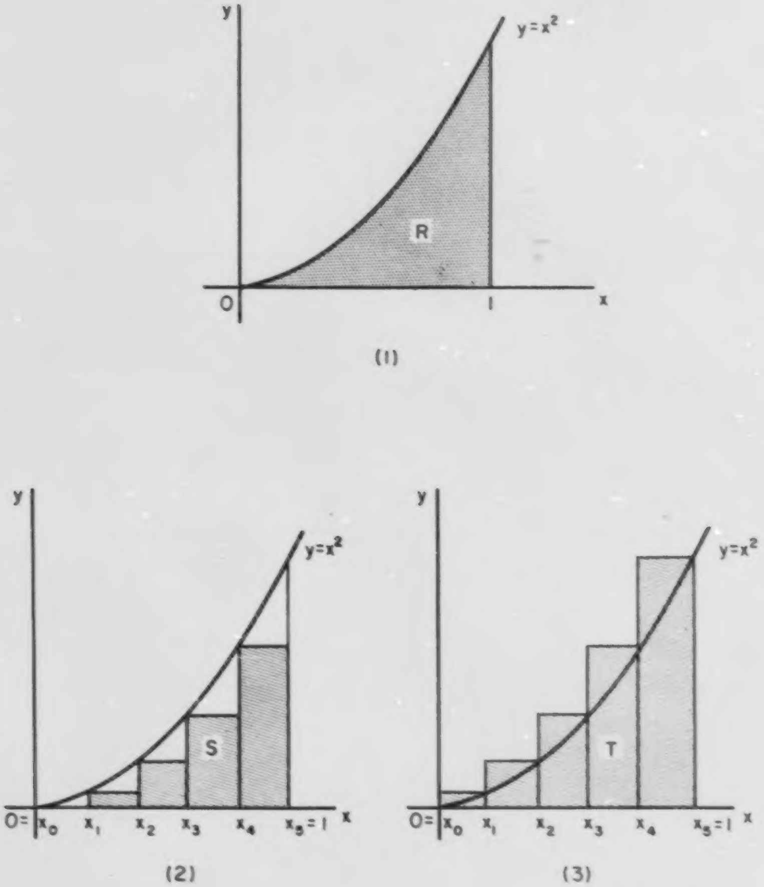
In abbreviated sum notation (Section A3-2) we have

$$\sum_{k=1}^{n} m_k(x_k - x_{k-1}) \leq \alpha(R) \leq \sum_{k=1}^{n} M_k(x_k - x_{k-1}) .$$

In Section 1-2 we were able to represent the area of the standard region under a curve as a limit of sums of areas of rectangles and we were able to estimate that limit from above and below within a given tolerance of error. So far we lack means for evaluating such limits in simpler terms. Here we show for a simple nonlinear case, the function $f : x \quad x^2$ on $[0,1]$. how to obtain such an evaluation using special summation techniques. In Sections A3-2 and A6-1 we show how areas of regions under other graphs can be evaluated by summation techniques; in so doing we are only demonstrating that a direct attack on the problem of evaluating area is feasible and that we are not compelled to use the subtler, but often simpler, methods developed in Chapters 7 and 10.

Consider the region R under the graph $y = x^2$ on $[0,1]$, (the shaded region in Figure 6-2b(1)). Since f is an increasing function on $[0,1]$ it will be easy to approximate $\alpha(R)$ from above and below in the manner of Section 1-2.

We use a subdivision of $[0,1]$ into n equal intervals by means of the subdivision points $x_0 = 0$, $x_1 = \frac{1}{n}$, ... , $x_{n-1} = \frac{n-1}{n}$, $x_n = \frac{n}{n} = 1$. On the k-th interval of the subdivision, $x_{k-1} \leq x \leq x_k$, we have $f(x_{k-1}) \leq f(x) \leq f(x_k)$ since f is increasing.


y
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x
y
y=x²
T
O=x₀
x₁
x₂
x₃
x₄
x₅=1
x

(1)

(2)

(3)

Figure 6-2b

We conclude that the standard region R_k based on the interval $[x_{k-1},x_k]$ contains the rectangle S_k of height $f(x_{k-1})$ and is contained in the rectangle T_k of height $f(x_k)$, both on the same base. The union of the non-overlapping rectangles S_k forms a region S which is contained within R , and the union of the rectangles T_k contains R . From the properties of area we may then obtain upper and lower estimates for the area $\alpha(R)$.

We have $\alpha(S) \leq \alpha(R) \leq \alpha(T)$, where

$$\begin{aligned}
\alpha(S) &= \sum_{k=1}^{n} f(x_{k-1})(x_k - x_{k-1}) \\
&= \sum_{k=1}^{n} \left(\frac{k-1}{n}\right)^2 \frac{1}{n} \\
&= \frac{1}{n^3}\sum_{k=1}^{n} (k^2 - 2k + 1) \\
&= \frac{1}{n^3}\left\{\sum_{k=1}^{n} k^2 - \sum_{k=1}^{n} (2k - 1)\right\}
\end{aligned}$$

and

$$\begin{aligned}
\alpha(T) &= \sum_{k=1}^{n} f(x_k)(x_k - x_{k-1}) \\
&= \frac{1}{n^3}\sum_{k=1}^{n} k^2 .
\end{aligned}$$

We recognize the second sum in the braces within the formula for $\alpha(S)$ as the sum of an arithmetic progression, the first n odd natural numbers, whose sum is n^2 . The sum $\sum_{k=1}^{n} k^2$ of the first n squares appears in both the formula for $\alpha(S)$ and that for $\alpha(T)$. A general treatment of such sums is given in Section A3-2. For this particular sum we have (Example A3-1g)

$$S_n = \sum_{k=1}^{n} k^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} .$$

Consequently,

$$\alpha(S) = \frac{1}{n^3}[\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} - n^2] = \frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2} ,$$

$$\alpha(T) = \frac{1}{n^3}[\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}] = \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2} .$$

Since S is contained in R, and R is contained in T, Property 2 of area states that

$$\alpha(S) \leq \alpha(R) \leq \alpha(T) ,$$

or

$$\frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2} \leq \alpha(R) \leq \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2} .$$

As we increase the number n of subdivisions, both $\alpha(S)$ and $\alpha(T)$ become steadily better approximations to the number $\frac{1}{3}$, and we conclude that $\alpha(R) = \frac{1}{3}$. Formally, given any tolerance $\epsilon > 0$ we choose n to satisfy the inequality

$$\frac{1}{2n} + \frac{1}{6n^2} < \epsilon ;$$

then $\alpha(R)$ differs from $\alpha(S)$ or $\alpha(T)$ by at most ϵ, and the estimate $\alpha(S)$ from below and $\alpha(T)$ from above differ from each other by at most 2ϵ.

Special summation techniques can be used to obtain the areas of standard regions for other functions. In Section A6-1 such summation techniques are used for the power function $x \longrightarrow x^n$ and the circular function $x \longrightarrow \cos x$. Often it is not convenient, sometimes not possible, to represent the area as a limit of sums which may be easily evaluated. The calculus offers simpler and more general techniques (Chapter 10) but these, too, may fail. The idea of approximation is the fundamental one, and if all else fails we can always resort to obtaining approximations from above and below to the area of a standard region.

Exercises 6-2

1. Use the summation method to find the area of the standard region defined by
   (a) $f : x \longrightarrow c$, $0 \leq x \leq b$, $c > 0$.
   (b) $f : x \longrightarrow cx$, $0 \leq x \leq b$, $c > 0$.
   (c) $f : x \longrightarrow x^2 + 2x$, $0 \leq x \leq b$.
   ^(d) $f : x \longrightarrow \sin (ax + b)$; $0 \leq x \leq c$; a , b , c such that $\sin (ax + b) \geq 0$ on $[0,c]$.
   ^(e) $f : x \longrightarrow \cos^2 x$, $0 \leq x \leq c$.

2. The problem posed in Section 1-2 was to determine the area of the standard region for $f : x \longrightarrow \sqrt{x}$ on $[0,1]$. The summation encountered there was similar to the one encountered in this section. Use this fact to solve the problem of Section 1-2.

3. Obtain the result of Exercise 2 using only the fact that the area under the graph of $f : x \longrightarrow x^2$ on $[0,1]$ is $\frac{1}{3}$, together with the basic properties of area, without resort to summation techniques.

4. Show how the upper estimating sums for $\sqrt{x}$ are related term-by-term to the lower estimating sums for x^2 . (Hint: Sketch a graph of $y = x^2$. Use this graph and the y-axis to represent the standard region defined by $\sqrt{x}$.)

5. If $S_n = \sqrt{1} + \sqrt{2} + \ldots + \sqrt{n}$, show that

$$\frac{2}{3}\sqrt{n^3} < S_n < \frac{2}{3}\sqrt{n^3} + \sqrt{n} .$$

6-3. The Concept of Integral. Integrals of Monotone Functions.

(i) Definition of integral.

In the computation of the area of the standard region under the graph of a bounded function f on a closed interval we gave upper and lower estimates of the area in terms of upper and lower bounds for f on each interval of a subdivision. If the function f takes on maximum and minimum values on each subinterval, as it would if f were continuous or monotone, then these would give the sharpest possible bounds. When f is continuous it may be easier to use slacker bounds than to attempt to determine the extrema. For monotone functions, however, the situation is especially simple: The extreme values on an interval are taken on at the endpoints.

We may allow f to take on negative values so that the interpretation of the upper and lower sums as upper and lower estimates of an area may not be immediate. Still these upper and lower sums may serve as upper and lower estimates for some unique number which lies below all upper estimates and above all lower estimates; if such a unique number exists it is called the integral of f over the base interval. The idea of integral has far-reaching applications, and its interpretation as area, although useful for visualizing the concept of integral, is not necessarily the most important realization of the concept.

We consider a bounded function f defined on a closed interval $[a,b]$. A subdivision of $[a,b]$ into n intervals is defined by a set of points

$$\sigma = \{x_0 , x_1 , x_2 , \ldots , x_{n-1} , x_n\}$$

where $x_0 = a$, $x_n = b$ and

$$x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n .$$

We shall call a set σ of points satisfying these requirements a partition of $[a,b]$. On the k-th subinterval $[x_{k-1},x_k]$ defined by the partition σ , let m_k be a lower bound, M_k an upper bound for $f(x)$, so that

$$m_k \leq f(x) \leq M_k$$

for all x in the subinterval. We define the lower sum over σ for the lower bounds m_k as

$$L = \sum_{k=1}^{n} m_k(x_k - x_{k-1})$$

and the upper sum over σ for the upper bounds M_k as

$$U = \sum_{k=1}^{n} M_k(x_k - x_{k-1}) .$$

If f is a nonnegative function then the lower and upper sums correspond to lower and upper estimates, respectively, for the area under the graph of f on $[a,b]$. More generally, without restricting the sign of f, we use the lower and upper sums to define the integral of f, if it exists.

DEFINITION 6-3. Let f be defined on $[a,b]$. We say that the number I is the integral of f over $[a,b]$ if there exists just one number I such that for each choice of partitions σ_1, σ_2 and all lower sums L_1 over σ_1 and upper sums U_2 over σ_2, we have

$$L_1 \leq I \leq U_2 .$$

We raise the question of existence of such a number I because it is not immediately clear. It is possible to prove that no lower sum is greater than any upper sum. Still, there may be a gap separating the values of the upper sums from those of the lower sums. If so, there is more than one number between the lower and upper sums and the integral is not defined. (This is true of the function f of Section 6-1. See Exercises A6-2, No. 4.) On the other hand, if for each $\epsilon > 0$ it is possible to find lower and upper sums which differ by less than ϵ, there is such a number I which these lower and upper sums approximate within the error tolerance ϵ; in other words, we are able to define I as the limit of upper and lower sums. We leave the proof of existence (under appropriate conditions) to the appendix (Section A6-2) but state the principal result here as a theorem which we shall use.

THEOREM 6-3a. Let f be a bounded function on $[a,b]$. If for every positive ϵ there exists a partition σ of $[a,b]$ and lower and upper sums L and U over σ which differ by less than ϵ, then there exists a number I which is the integral of f over $[a,b]$. Conversely, if f is integrable over $[a,b]$ then there exists a partition σ with lower and upper sums L and U such that $U - L < \epsilon$.

If f has an integral I over $[a,b]$ we say that f is integrable over $[a,b]$.

A proof of Theorem 6-3a requires a verification of the conditions of Definition 6-3. First we must have a demonstration that no upper sum is less than any lower sum (Lemma A6-2b). In that event, there exists at least one number which is both a lower bound for the set of upper sums and an upper bound for the set of lower sums (Separation Axiom, Appendix 1-5). It must then be shown that there is at most one number I between the upper and lower sums. This follows from the existence of an upper and a lower sum which are closer together than any prescribed tolerance ϵ (Lemma A1-5). Thus the integral is determined by a squeeze between upper and lower sums. For the details see Section A6-2.

(ii) Integrability of monotone functions.

For monotone functions we may choose m_k and M_k as function values at the endpoints of $[x_{k-1},x_k]$ and it is particularly easy to obtain an estimate of the difference between the upper and lower the error of approximations to the integral. We picture the situation in terms of the area of a standard region for a nonnegative increasing function f.

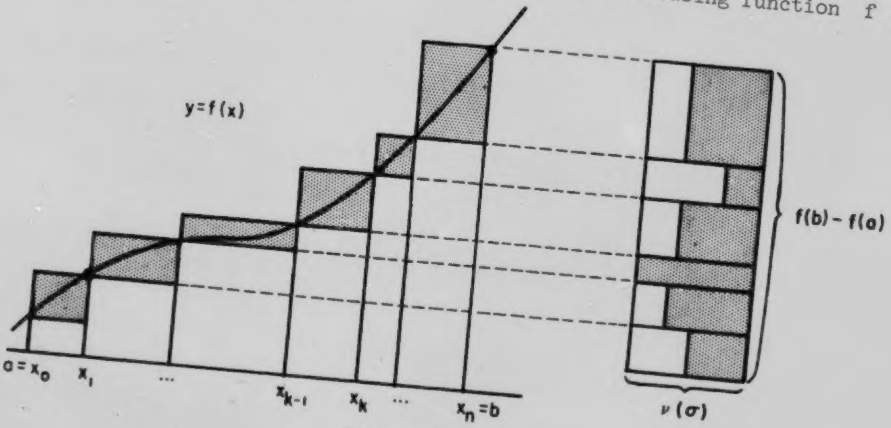

Figure 6-3a

In Figure 6-3a, the shaded rectangle over the interval $[x_{k-1},x_k]$ has height $M_k - m_k$, where $M_k = f(x_k)$ and $m_k = f(x_{k-1})$.

The total area of the shaded rectangles is the difference between the upper and lower sums for the given partition.

Since the function f is monotone we can imagine sliding these rectangles parallel to the x-axis into an arrangement with their right sides aligned. In this arrangement the rectangles are contained without overlapping in a single rectangle of height $f(b) - f(a)$ and base equal to the length of the largest interval of the subdivision. The length of the largest interval,

$$\nu(\sigma) = \max\{x_k - x_{k-1}\} ,$$

is a measure of the coarseness of the subdivision and is called the norm of the partition σ. We have depicted a bound on the difference between the upper and lower sums:

$$U - L \leq [f(b) - f(a)]\nu(\sigma) .$$

Clearly, we can make the difference between U and L less than any error tolerance ϵ by making the subdivision fine enough, namely, by choosing σ so that

$$\nu(\sigma) < \frac{\epsilon}{f(b) - f(a)} .$$

Since the area I must then lie in the interval of length at most ϵ between U and L its value cannot differ from either by more than ϵ and we have satisfied the condition of Theorem 6-3a.

Although we have obtained the last result by a geometrical argument we can obtain the same result analytically with more generality: any function monotone on a closed interval is integrable.

THEOREM 6-3b. If f is monotone on $[a,b]$, then f is integrable over $[a,b]$.

Proof: We show that for each positive ϵ it is possible to find a partition σ of $[a,b]$ for which the difference between the upper and lower sums on the partition can be made less than ϵ:

$$U - L < \epsilon .$$

For this purpose we let M_k be the maximum and m_k the minimum of f on $[x_{k-1},x_k]$. We shall prove that it is sufficient to use a subdivision σ with a norm satisfying

$$\nu(\sigma) < \frac{\epsilon}{|f(b) - f(a)|}$$

when $f(b) \neq f(a)$.

The case $f(b) = f(a)$ is trivial since the function f must then be a constant function. In this case, we have $M_k = m_k$ and

$$U - L = 0$$

for all subdivisions σ .

We consider the case of a weakly increasing function f (the weakly decreasing case is similar). The maximum and minimum on $[x_{k-1}, x_k]$ are given by the endpoint values

$$M_k = f(x_k) \quad \text{and} \quad m_k = f(x_{k-1}) .$$

Summing over the intervals of the subdivision we have

$$U = \sum_{k=1}^{n} M_k(x_k - x_{k-1}) = \sum_{k=1}^{n} f(x_k)(x_k - x_{k-1})$$

$$L = \sum_{k=1}^{n} m_k(x_k - x_{k-1}) = \sum_{k=1}^{n} f(x_{k-1})(x_k - x_{k-1}) .$$

Consequently,

$$\begin{aligned} U - L &= \sum_{k=1}^{n} [f(x_k) - f(x_{k-1})](x_k - x_{k-1}) \\ &\leq \sum_{k=1}^{n} [f(x_k) - f(x_{k-1})]\nu(\sigma) \\ &\leq \nu(\sigma) \sum_{k=1}^{n} [f(x_k) - f(x_{k-1})] . \end{aligned}$$

We observe that

$$\sum_{k=1}^{n} f(x_k) = f(x_1) + f(x_2) + \ldots + f(x_{n-1}) + f(x_n)$$

and

$$\sum_{k=1}^{n} f(x_{k-1}) = f(x_0) + f(x_1) + \dots + f(x_{n-1}) .$$

Subtracting the second of these sums from the first, we have

$$\sum_{k=1}^{n} [f(x_k) - f(x_{k-1})] = f(x_n) - f(x_0) = f(b) - f(a) ;$$

consequently,

$$U - L \leq \nu(\sigma)[f(b) - f(a)] .$$

To make the difference less than ϵ we need only chose $\nu(\sigma)$ as indicated above. We have satisfied the condition of Theorem 6-3a and it follows that f is integrable over $[a,b]$.

(iii) *Riemann sums*. *Notation*.

We have employed a method for defining area by approximation from above and below and extended our approach to define the more general concept of integral. This method has the great advantage of logical simplicity in the derivation of properties of the integral.

A more direct method, but one which requires somewhat more complicated argument, is to utilize values of the function in the intervals of a subdivision, instead of upper and lower bounds for approximating the area. Thus, for a function f defined on $[a,b]$ and a partition $\sigma = \{x_0 , x_1 , x_2 , \dots , x_n\}$ of $[a,b]$ we introduce sums of the form

$$R = \sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}) \tag{1}$$

where ξ_k is any value in the subinterval $[x_{k-1},x_k]$. These are called *Riemann sums* *. For a general Riemann sum the rectangle over $[x_{k-1},x_k]$ will usually not include all of the standard region under the graph and will usually include some region above the curve (Figure 6-3b) so that there will be a partial cancellation of errors. Since $m_k \leq f(\xi_k) \leq M_k$, no matter how ξ_k is chosen, we see that the Riemann sums are sandwiched between the upper and lower sums

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*After Bernhard Riemann, a German mathematician of the early 19th century, a pioneer in the careful study of the concept of integral and in other important areas.

$$L \leq R \leq U .$$

If f has an integral I, we can therefore approximate I by Riemann sums. In fact, the approximation to I by Riemann sums can be kept within any prescribed tolerance of error for every sufficiently fine subdivision σ and corresponding choice of ξ_k (Section A6-2). We shall then have determined the integral as a new kind of limit, a limit of Riemann sums:

$$\lim_{\nu(\sigma)\sim 0} R .$$

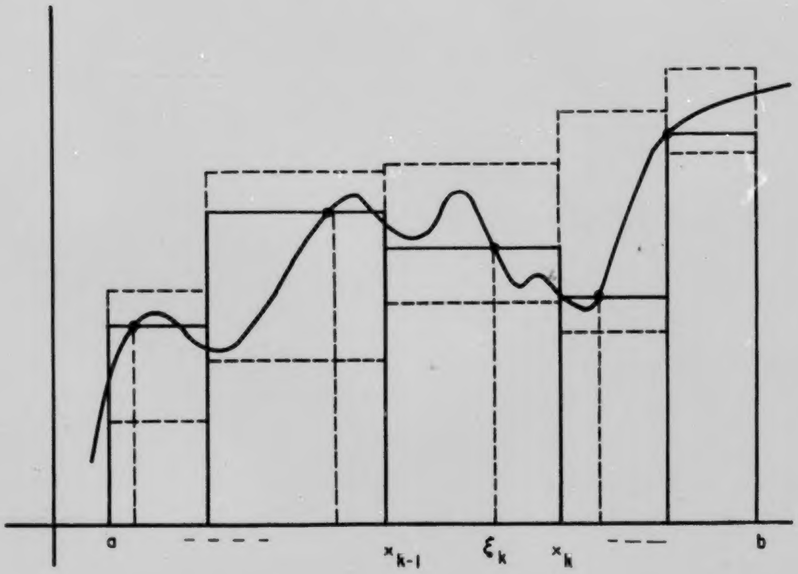

Figure 6-3b

It is natural to suppose that if this limit of Riemann sums exists, then so does the integral I of Definition 6-3, and to suppose that the two are the same. This is not an obvious proposition[1], but it is true (Exercises A6-2, No. 3). These remarks are summarized in the following theorem.

THEOREM 6-3c. The value I is the integral of f over [a,b], in the sense of Definition 6-3, if and only if it is the limit of Riemann sums,

$$I = \lim_{\nu(\sigma)\sim 0} R .$$

The proof is left to Section A6-2.

The integral I of f over [a,b] is usually written in the elegant notation of Leibniz. In Leibnizian notation, the Riemann sum (1) is written

$$R = \sum_{k=1}^{n} f(\xi_k)\Delta x_k$$

where Δx_k represents the difference $x_k - x_{k-1}$. In representing the integral Leibniz used a form reminiscent of the Riemann sums,

$$I = \int_a^b f(x)\, dx .$$

We shall call the endpoints a and b of the interval of integration, the lower and upper ends of integration, respectively[2].

Although, as we shall see, the Leibnizian notation for integral nicely complements the Leibnizian notation for derivative, it stems from conceptions which cannot abide the light of logical reason. In the thinking of Leibniz and most of the early users of the calculus, the integral sign $\int$ which is an elongated Roman "S" is a special summation symbol which replaces the corresponding Greek symbol "Σ". The integral $\int_a^b f(x)\, dx$ was thought of as the sum of the areas of the infinite set of "rectangles" having "infinitesimal" or "immeasurably small" base dx and height f(x) for

---

[1] There are concepts of integral which are more general than the one we consider here. For example the functions of Exercises 3-5, No. 12 are integrable in a more general context, but not in the sense of Riemann.

[2] It is customary to call these values the lower and upper limits or bounds of integration, but the terms "limit" and "bound" are used so often in other senses in the discussion of integral that we choose to break with tradition and introduce a new term.

$a \leq x \leq b$ (the Roman "d" in "dx" replaces the Greek "Δ" of the finite Riemann sum). These ideas are nonsensical on their face[1], as the redoubtable metaphysician Berkeley made plain to his contemporaries.

Only Newton among the mathematicians of his age had some slight success in clarifying the basic limit concepts involved, and even he used the idea of infinitesimal freely when it suited his purpose.[2] The task of providing a logical foundation for the calculus was effectively begun by mathematicians of the nineteenth century. Nonetheless, the idea of summation of "infinitesimals" was both suggestive and fruitful. In ancient times, Archimedes in "The Method" made ingenious use of it to discover (not prove) formulas for the areas of conic sections. Euler used this nonsense without question and managed to develop vast areas of analysis without a clear-cut definition of limit. On the other hand, the imprecise ideas of Leibniz and his contemporaries have their pitfalls and mathematicians were not always successful in avoiding them.

(iv) Arclength

We have already made some use of particular Riemann sums. In estimating the integral of a monotone function we used upper and lower Riemann sums formed by taking as bounds the maximum and minimum values of f in each interval of a subdivision. We could also use upper and lower Riemann sums for the continuous functions since they share with the monotone functions the property of having a maximum and minimum value on each closed interval (Theorem 3-7b). For continuous functions, however, the estimates by upper and lower Riemann sums are not the appropriate general tool because the extrema of a continuous function on an arbitrary closed interval may be beyond simple analysis, as would be the case for the Weierstrass function of Section A4-3. In other cases, the Riemann sum may be the appropriate device because it is inherently impossible to obtain the necessary bounds on every subinterval, as the following example shows.

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[1] The concept of "infinitesimal" requires the existence of quantities which are smaller in absolute value than any positive number and yet not zero. For real numbers such a conception is inherently self-contradictory (see Exercises A1-3, No. 13b).

[2] See Boyer, C.B. The Concepts of the Calculus, Columbia University Press. New York. 1939.

Example 6-3. The length of an arc of a continuous curve is another quantity which can be defined in terms of an integral. Given a continuous function f on $[a,b]$ and a partition σ of $[a,b]$ it is natural to attempt to approximate the length L of the arc of the graph between $x = a$ and $x = b$ by the length P of the polygonal arc joining the successive points of the graph corresponding to the partition points. For $y_k = f(x_k)$, $\Delta x_k = x_k - x_{k-1}$, and $\Delta y_k = y_k - y_{k-1}$, the length of this polygonal arc is

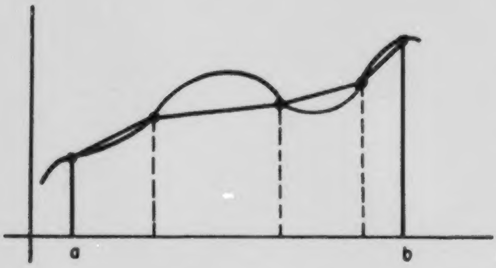

Figure 6-3c

$$P = \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} .$$

This sum can be put in the form of a Riemann integral. We observe if f is differentiable that, by the Law of the Mean, $\frac{\Delta_k y}{\Delta_k x} = f'(\xi_k)$ for some value ξ_k satisfying $x_{k-1} < \xi_k < x_k$. Consequently,

(1) $$P = \sum_{k=1}^{n} \sqrt{1 + f'(\xi_k)^2}\, \Delta x_k ;$$

that is, P is a Riemann sum for the function $g : x \longrightarrow \sqrt{1 + f'(x)^2}$. We then define the length L of the arc by the formula

(2) $$L = \int_a^b g(x)dx = \int_a^b \sqrt{1 + f'(x)^2}\, dx ,$$

if the integral exists.

There is one peculiarity of this treatment of arclength which you should know. Since the segment has the shortest length of all arcs joining two points (this is assumed here, but it can be proved), it follows on summing over the intervals of a subdivision that (1) is always a lower sum for (2). Without an obvious geometrical way to describe corresponding upper sums, we must abandon our idea of approximation from above and below. Thus we are compelled to use the Riemann sum approach to arclength.

Exercises 6-3

1. Evaluate the integral of each function f over the indicated interval.

   (a) $f(x) = 2 - x^2$ $\quad 0 \leq x \leq 1$

   (b) $f(x) = x$ $\quad 1 \leq x \leq 2.5$

   (c) $f(x) = \frac{5}{2}$ $\quad 2.5 \leq x \leq 3$

   (d) $f(x) = 5 - x$ $\quad 3 \leq x \leq 5$

2. (a) Find the minimum and the maximum values of $f(x) = 2 + 2x - x^2$ on the interval $[0,1]$, and use them to find two numbers respectively below and above the value of $\int_0^1 f(x)\,dx$.

   (b) Check your result by evaluating the integral.

3. Find upper and lower sums differing by less than .1 for the area under the graph of $f : x \longrightarrow \frac{1}{x}$ on $[1,2]$.

4. Evaluate each of the following integrals.

   (a) $\int_{-1}^{1} x^3\,dx$.

   (b) $\int_{-2}^{2} |x|\,dx$.

   (c) $\int_{-1}^{1} x^2\,dx$. (See Exercises 6-4, No. 4)

5. Approximate $\int_0^1 \frac{1}{1 + x^2}\,dx$ by Riemann sums.

6. A function f defined on the interval $[a,b]$ is said to be a step-function on $[a,b]$ if for some partition $\sigma = \{x_0, x_1, \ldots, x_n\}$ of the interval, $f(x)$ is constant on each open subinterval (x_{k-1}, x_k), $k = 1, 2, \ldots, n$. Thus sgn x is a step function on $[-1,1]$, where sgn x is defined by

$$\operatorname{sgn} x = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 . \end{cases}$$

   (a) Prove that a step function is integrable.

   (b) Find $\int_a^b \operatorname{sgn} x\,dx$.

7. Evaluate each of the following integrals:

(a) $\int_{-1}^{3} [3x + 4]\, dx$. (c) $\int_{1}^{5} \sqrt{2[x]}\, dx$.

(b) $\int_{0}^{10} \left[\frac{x}{4}\right] dx$. (d) $\int_{1}^{5} \left[\sqrt{2x}\right] dx$.

8. Let $= \{x_0, x_1, x_2, \ldots, x_n\}$ be a partition of $[a,b]$, and f , a function which is integrable over each interval $[x_{k-1}, x_k]$, $k = 1 , 2 , \ldots , n$. Prove that f is integrable over the entire interval $[a,b]$ and that

$$\int_a^b f(x)dx = \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} f(x)dx .$$

9. (a) Prove for $0 < x_{k-1} < x_k$ that

$$x_{k-1}^2 (x_k - x_{k-1}) < \frac{x_k^3 - x_{k-1}^3}{3} < x_k^2 (x_k - x_{k-1}) .$$

(b) From Part (a) prove that for $0 < a < b$

$$\int_a^b x^2\, dx = \frac{b^3}{3} - \frac{a^3}{3} .$$

(c) Generalize this method to obtain

$$\int_a^b x^n\, dx$$

for $0 < a < b$.

6-4. Elementary Properties of Integrals.

(i) Geometrically suggested properties.

Beginning with the postulated properties of area in Section 6-1, we formulated the geometrical concept of area of a standard region in terms of the analytical concept of integral. The concept of integral is somewhat more general than that of area; not only may the integral be defined for negative functions, but the integral may also be defined for functions for which intuition suggests no interpretation for the area under the graph (Exercises A6-2, No. 5). Nonetheless, the postulated properties of area suggest properties true of integrals in general.

Let f and g be nonnegative functions with $f(x) \leq g(x)$ on $[a,b]$. Since the standard region under the graph of f is contained in the standard region under the graph of g (Figure 6-4a), from Property 2 of Section 6-1 the area of the former must be no greater than the area of the latter. A similar inequality holds for integrals in general.

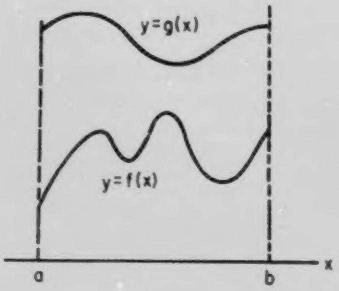

Figure 6-4a

THEOREM 6-4a. If f and g are integrable and $f(x) \leq g(x)$ on $[a,b]$ then

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx .$$

Proof. Let I denote the integral of f over $[a,b]$, and J the integral of g. We know (Theorem 6-3a) that for every positive ϵ there exist upper and lower sums U and L for g such that $U - L < \epsilon$. Since $L \leq J \leq U$ (Definition 6-3) we conclude that $U - J < \epsilon$. Thus we can find upper sums as close as desired to J. At the same time, every upper sum for J is an upper sum for I since $f(x) \leq g(x)$. We have $I \leq J$, for if we had $I > J$ we could take $\epsilon = I - J > 0$ and from $U - J < I - J$ it would follow that $U < I$, a contradiction, since U is an upper sum for I.

Consider the decomposition of the standard region over $[a,c]$ into the two standard regions over $[a,b]$ and $[b,c]$ where $a < b < c$ (see Figure 6-4b). The additive property of area (Property 3 of Section 6-1) states that the sum of the areas of the two subregions must be the area of the entire region. This corresponds to a general statement for integrals.

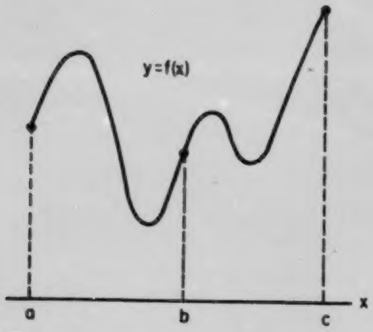

Figure 6-4b

THEOREM 6-4b. If f is integrable over $[a,c]$ then, for $a < b < c$,

$$(1) \qquad \int_a^b f(x)dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx \,.$$

Proof. From the integrability of f over $[a,c]$, it follows that f is integrable over the subintervals $[a,b]$ and $[b,c]$, (Lemma A6-2c). Consequently, for any $\epsilon > 0$ according to Theorem 6-3a we can find subdivisions σ' of $[a,b]$ and σ'' of $[b,c]$ with corresponding upper and lower sums, U' , L' and U'' , L'' such that

$$U' - L' < \epsilon \text{ and } U'' - L'' < \epsilon \,.$$

Clearly, $U = U' + U''$ and $L = L' + L''$ are upper and lower sums over $[a,c]$ for the partition σ constructed by taking the two partitions σ' and σ'' together as a partition of $[a,c]$. Furthermore,

$$U - L = (U' - L') + (U'' - L'') < 2\epsilon \,.$$

For the integrals I , I' , I'' over the intervals $[a,c]$, $[a,b]$, $[b,c]$, respectively, we have

$$U - I < 2\epsilon \,,\; U' - I' < \epsilon \,,\; U'' - I'' < \epsilon \,,$$

whence, for every positive ϵ ,

$$\begin{aligned} |I' + I'' - I| &= |(I' - U') + (I'' - U'') - (I - U)| \\ &< \epsilon + \epsilon + 2\epsilon \\ &\le 4\epsilon \,. \end{aligned}$$

It follows that $I' + I'' = I$, as we sought to prove.

Up to this point the symbol $\int_a^b f(x)dx$ has been defined only for $a < b$. We now define the integral so as to make (1) universally valid. Formally substituting a for b and c in (1) we obtain

$$\int_a^a f(x)dx + \int_a^a f(x)dx = \int_a^a f(x)dx$$

which can be satisfied only if $\int_a^a f(x)dx = 0$. We define the integral accordingly:

DEFINITION 6-4a. If a is any point of the domain of f , we define

$$\int_a^a f(x)dx = 0 .$$

Furthermore, if we formally set $c = a$ in (1), we obtain

$$\int_a^b f(x)dx + \int_b^a f(x)dx = \int_a^a f(x)dx = 0 .$$

This equation suggests the following definition.

DEFINITION 6-4b. If f is integrable over $[a,b]$, we define

$$\int_b^a f(x)dx = -\int_a^b f(x)dx .$$

With these definitions, Equation (1) becomes valid independently of the order of a , b , c .

Corollary. If a , b , and c are any points of an interval over which f is integrable, then

$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx .$$

The proof is left as an exercise.

Example 6-4a. Consider $\int_0^a x^2\,dx$ where $a > 0$. Since x^2 is monotone for $x > 0$ the integral exists. We consider a subdivision of $[0,a]$ into n equal parts and in the manner of Section 6-2, obtain upper and lower sums

$$U = \sum_{k=1}^{n} \left(\frac{ka}{n}\right)^2 \frac{a}{n} = \frac{a^3}{n^3} \sum_{k=1}^{n} k^2$$

and

$$L = \sum_{k=1}^{n} \left[\frac{(k-1)a}{n}\right]^2 \frac{a}{n} = U - \frac{a^3}{n} .$$

We utilize the same summation formula as in Section 6-2 and obtain by the same arguments,

$$\int_0^a x^2\,dx = \frac{a^3}{3} .$$

This is a general formula, valid for all positive values a (and negative values also, Exercises 6-4, No. 2). Now, applying the Corollary to Theorem 6-4b, we can obtain the integral of f between any positive ends of integration whatever:

$$\begin{aligned}\int_a^b f(x)dx &= \int_a^0 f(x)dx + \int_0^b f(x)dx \\ &= \int_0^b f(x)dx - \int_0^a f(x)dx \\ &= \frac{b^3}{3} - \frac{a^3}{3} .\end{aligned}$$

Example 6-4b. Property 4 of area, that the area of a rectangle is the product of the lengths of two adjacent sides, tells us for $f(x) = c$ where c is a positive constant, that the area of the standard region on $[a,b]$ is $c(b - a)$, (Figure 6-4c). More generally, whether c is positive or not, and no matter what the values of a and b, it is true that

$$\int_a^b c\,dx = c(b - a) .$$

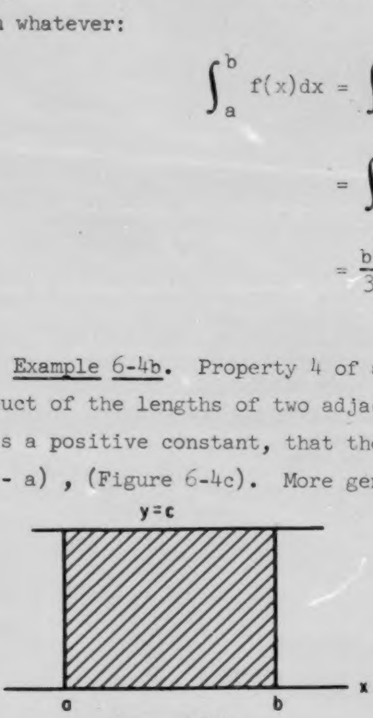

Figure 6-4c

We need only prove the result for $a < b$; if $a \geq b$ the result follows by the Corollary to Theorem 6-4b.

In every interval, c is both an upper bound and a lower bound for $f(x) = c$. For every partition of $[a,b]$, then, we take

$$\begin{aligned} U = L &= \sum_{k=1}^{n} c(x_k - x_{k-1}) \\ &= c\sum_{k=1}^{n} (x_k - x_{k-1}) \\ &= c(b - a) . \end{aligned}$$

Example 6-4c. Consider the area under the graph of $f(x) = x$ on $[a,b]$ $a \geq 0$, (Figure 6-4d). This region is a trapezoid with parallel bases of length a and b and altitude $b - a$. We know from elementary geometry

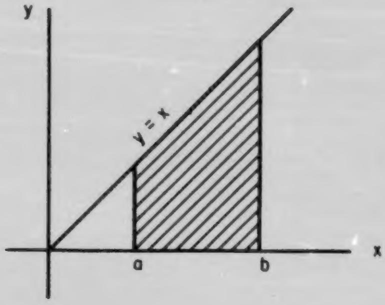

Figure 6-4d

that the area of such a trapezoid is

$$\frac{1}{2}(b - a)(b + a) = \frac{b^2}{2} - \frac{a^2}{2} .$$

More generally, we prove for all a and b that

$$\int_a^b x\, dx = \frac{b^2}{2} - \frac{a^2}{2} .$$

Again, it is sufficient to prove the result for $b > a$; for other cases the result follows from Definitions 6-4a and 6-4b. We subdivide the interval $[a,b]$ into n equal parts and obtain upper and lower sums

$$U = \sum_{k=1}^{n} (a + kh)h$$

$$L = \sum_{k=1}^{n} [a + (k - 1)h]h = U - nh^2$$

where $h = \frac{b - a}{n}$. We have from the rule for summation of an arithmetic progression,

$$\begin{aligned} U = h\sum_{k=1}^{n} (a + kh) &= \frac{nh}{2}[2a + (n + 1)h] \\ &= \frac{b - a}{2}[2a + (b - a) + h] \\ &= \frac{b^2}{2} - \frac{a^2}{2} + \frac{(b - a)h}{2} . \end{aligned}$$

For the lower sum, then

$$L = U - nh^2 = U - (b - a)h = \frac{b^2}{2} - \frac{a^2}{2} - \frac{(b - a)h}{2} .$$

Combining these results and taking the limit as n approaches infinity we obtain the anticipated value for the integral.

(ii) Linearity of integration.

For positive constants α and β integration is a linear operation:

$$\int_a^b [\alpha f(x) + \beta g(x)]dx = \alpha\int_a^b f(x)dx + \beta\int_a^b g(x)dx ,$$

for if U' and L' are upper and lower sums for f , U'' and L'' for g , it is immediate that $U = \alpha U' + \beta U''$ and $L = \alpha L' + \beta L''$ are upper and lower sums for the linear combination $\alpha f(x) + \beta g(x)$. This result does not depend on the signs of α and β as we now prove.

THEOREM 6-4c. If f and g are integrable over $[a,b]$ then any linear combination $\alpha f + \beta g$ is integrable over $[a,b]$ and

$$\int_a^b [\alpha f(x) + \beta g(x)]dx = \alpha\int_a^b f(x)dx + \beta\int_a^b g(x)dx .$$

To simplify the considerations which depend on the signs of α and β we divide the proof into two parts.

Lemma 6-4a. If f is integrable over $[a,b]$ then for any constant α, the function αf is integrable and

$$\int_a^b \alpha f(x)dx = \alpha \int_a^b f(x)dx .$$

Proof. Let σ be a partition of $[a,b]$ and take upper and lower sums over σ,

$$U = \sum_{k=1}^{n} M_k(x_k - x_{k-1})$$

$$L = \sum_{k=1}^{n} m_k(x_k - x_{k-1}) ,$$

for which $U - L < \epsilon$.

If $\alpha > 0$, then

$$\alpha U = \sum_{k=1}^{n} \alpha M_k(x_k - x_{k-1}) \quad \text{and} \quad \alpha L = \sum_{k=1}^{n} \alpha m_k(x_k - x_{k-1})$$

are upper and lower sums, respectively, for αf. It follows that

$$\alpha U - \alpha L < \alpha\epsilon$$

and hence that the difference between upper and lower sums for αf can be made less than any desired tolerance. It follows that αf is integrable. Furthermore, for the integral I of f and J of αf over $[a,b]$ we have

$$U - I < \epsilon \quad , \quad \alpha U - J < \alpha\epsilon$$

from which it follows that

$$\begin{aligned} |J - \alpha I| &= |(J - \alpha U) + \alpha(U - I)| \\ &\leq |J - \alpha U| + \alpha|U - I| \\ &< 2\alpha\epsilon . \end{aligned}$$

Since this result holds for all positive ϵ, we conclude that $J = \alpha I$.

If $\alpha < 0$ then αU is a lower sum and αL an upper sum for αf. The proof is thus reduced to the preceding (Exercises 6-4, No. 3).

If $\alpha = 0$, the lemma follows trivially.

We have not attempted to give an interpretation of the integral in terms of area for functions which take on negative values. If $f(x) \leq 0$ on $[a,b]$ then $-f(x) \geq 0$. We have, in the light of Lemma 6-4a,

$$\int_a^b f(x)dx = -\int_a^b [-f(x)]dx \ .$$

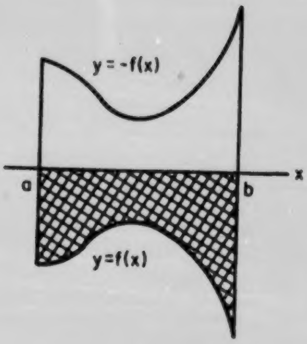

Figure 6-4e

The integral I of f over $[a,b]$ is, therefore, the negative of the area of the standard region under the graph of $-f$ (Figure 6-4e). Alternatively, we may consider I as the negative of the area of the region

$$\{(x,y) : a \leq x \leq b \ , \ 0 \geq y \geq f(x)\} \ ,$$

the region bounded below by the graph of f and above by the x-axis.

In general the integral may be interpreted as the signed area between the graph of f and the x-axis, where the signed area is positive and equal to the area under the graph for the part of the graph above the x-axis, and where the signed area is negative and equal to the negative of the area between the graph and x-axis for the part below. In particular, if the graph of f is symmetric with respect to the origin we have $f(-x) = -f(x)$ and for any interval $[-a,a]$ centered at the origin the signed area of any standard region above the x-axis on one side of the origin is the negative of the signed area of the symmetrically situated region below the x-axis on the other side of the origin; in this case $\int_{-a}^{a} f(x)dx = 0$. (Exercises 6-4, No. 4.)

Lemma 6-4b. If f and g are integrable over $[a,b]$, then $f + g$ is integrable over $[a,b]$ and

$$\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx \ .$$

We make use of an auxiliary result (Lemma A6-2d): Given any fixed tolerance, for any integrable function all sufficiently fine partitions have upper and lower sums closer then that tolerance. Thus for each positive ϵ, there exists some δ such that any partition σ will have an upper sum U and a lower sum L satisfying

$$|U - L| < \epsilon$$

whenever

$$\nu(\sigma) < \delta .$$

Let δ_1 and δ_2 be the controls corresponding to the given ϵ for f and g, respectively, and take $\delta = \min\{\delta_1, \delta_2\}$. Let σ be any partition with $\nu(\sigma) < \delta$. There then exist upper and lower sums over σ, U' and L' for f, and U'' and L'' for g such that

$$|U' - L'| < \epsilon \quad \text{and} \quad |U'' - L''| < \epsilon .$$

Recall that

$$U' = \sum_{k=1}^{n} M_k'(x_k - x_{k-1}) , \quad L' = \sum_{k=1}^{n} m_k'(x_k - x_{k-1})$$

and

$$U'' = \sum_{k=1}^{n} M_k''(x_k - x_{k-1}) , \quad L'' = \sum_{k=1}^{n} m_k''(x_k - x_{k-1})$$

where

$$m_k' \leq f(x) \leq M_k' \quad \text{and} \quad m_k'' \leq g(x) \leq M_k'' .$$

Since

$$m_k' + m_k'' \leq f(x) + g(x) \leq M_k' + M_k''$$

it follows that $U = U' + U''$ is an upper sum and $L = L' + L''$ a lower sum, for $f + g$ over σ. We conclude that

$$U - L = (U' - L') + (U'' - L'') < 2\epsilon ,$$

and it follows that $f + g$ is integrable. Furthermore, for the integrals I', I'' and I of f, g and $f + g$, respectively, we have the estimate

$$\begin{aligned} |I' + I'' - I| &= |(I' - U') + (I'' - U'') - (I - U)| \\ &\leq |I' - U'| + |I'' - U''| + |I - U| \\ &< \epsilon + \epsilon + 2\epsilon \\ &< 4\epsilon \end{aligned}$$

for each positive ϵ. It follows that $I = I' + I''$.

The derivation of Theorem 6-4c from the preceding lemmas is simple and is left as an exercise.

In Examples 6-4a, b , c we have shown how to integrate x^2 , a constant, and x . Employing Theorem 6-4c we now have the means to integrate any quadratic function without further resort to estimates by upper and lower sums:

$$\int_a^b (Ax^2 + Bx + C)\,dx = A\int_a^b x^2\,dx + B\int_a^b x\,dx + C\int_a^b dx \,.$$

An immediate application of Theorem 6-4c gives the area between the graphs of two functions f and g on $[a,b]$, where $f(x) \leq g(x)$, as the integral of their difference. If $f(x) \geq 0$ as in Figure 6-4a then the area between the two graphs is simply the area of the standard region under the graph of g less the area of the standard region under the graph of f , that is,

$$\int_a^b g(x)\,dx - \int_a^b f(x)\,dx = \int_a^b [g(x) - f(x)]\,dx \,.$$

There is no reason to restrict these considerations to nonnegative functions, for if $f(x) < 0$ for some x in $[a,b]$, and m is a lower bound of $f(x)$ on $[a,b]$, we translate the x-axis vertically $|m|$ units in the negative direction so that

$$(x,y) \longrightarrow (x, y + |m|) \,.$$

In the new coordinate system the region lies between the graphs of the nonnegative functions $\overline{f} : x \longrightarrow f(x) + |m|$ and $\overline{g} : x \longrightarrow g(x) + |m|$. (Figure 6-4f.)

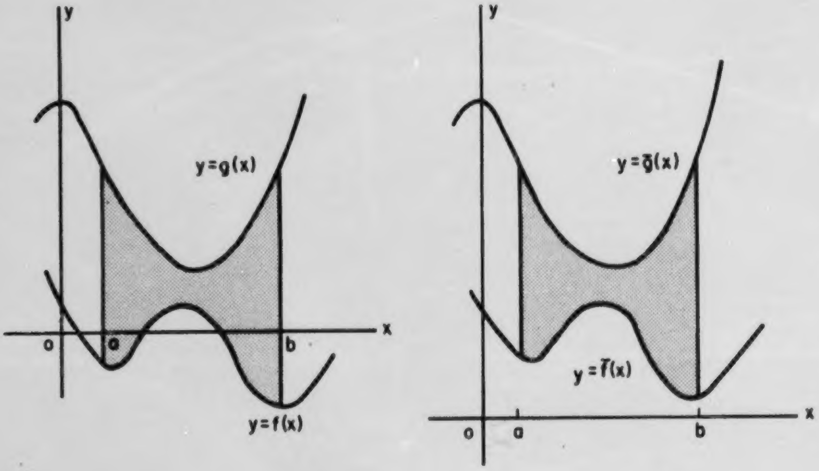

Figure 6-4f

Since $\bar{g}(x) - \bar{f}(x) = g(x) - f(x)$ the definition of the area of the region between the graphs of f and g as the integral of the function $g - f$ is clearly appropriate whenever $f(x) \leq g(x)$ on $[a,b]$. Thus, the area of the standard region under the graph of $F : x \longrightarrow g(x) - f(x)$ on $[a,b]$ (Figure 6-4g) is equal to the area of the region between the graphs of f and g on $[a,b]$ (Figure 6-4f).

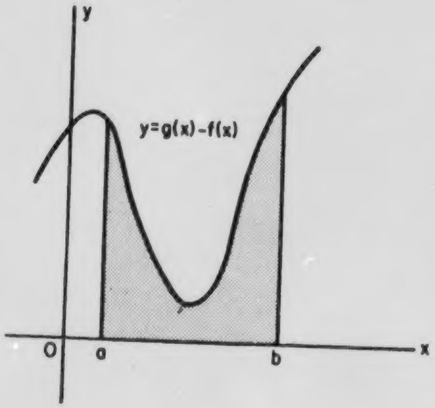

Figure 6-4g

Example 6-4d. Consider the area of the region between the graphs of the functions $f : x \longrightarrow \cos^2 x$ and $g : x \longrightarrow -\sin^2 x$ on $[0,4]$. (Figure 6-4h.)

We might attempt to represent the area of the region as the limit of sums of areas of rectangles. On the other hand, we know that the area is given by

$$\int_0^4 [f(x) - g(x)]dx ,$$

since $f(x) \geq g(x)$ for all x in the interval $[0,4]$.

But $$\int_0^4 [f(x) - g(x)]dx = \int_0^4 dx = 4 ,$$ since $f(x) - g(x) = \cos^2 x - (-\sin^2 x) = 1$ for all x. (The graph of $F : x \longrightarrow f(x) - g(x)$ is shown in Figure 6-4i.) In conclusion we note that the area of the region shaded in Figure 6-4i is equal to the area of the region shaded in Figure 6-4h.

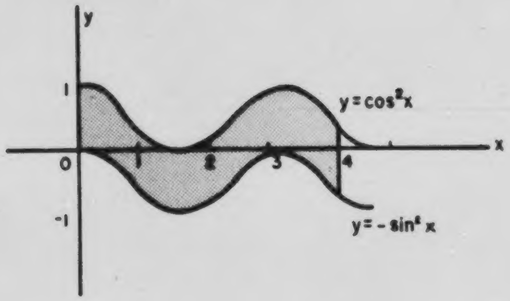

Figure 6-4h

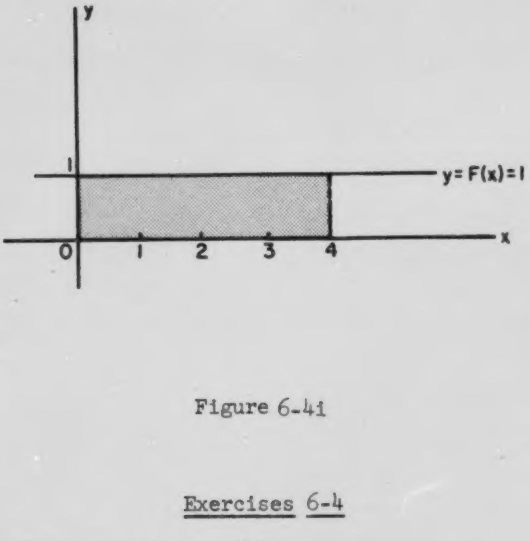

Figure 6-4i

Exercises 6-4

1. Prove the corollary to Theorem 6-4b.

2. (a) Show that $f : x \longrightarrow x^2$ is integrable over $[a,b]$, no matter what the values of a and b .

   (b) Prove that $\int_0^a x^2\,dx = \frac{a^3}{3}$ for $a \leq 0$.

   (c) Prove that $\int_a^b x^2\,dx = \frac{b^3}{3} - \frac{a^3}{3}$ for all values of a and b .

3. Exhibit the details of the proof of Lemma 6-4a when $\alpha < 0$.

4. (a) If the graph of f is symmetric with respect to the origin, then f is odd. Prove that if f is odd and integrable on $[-a,a]$, then

$$\int_{-a}^{a} f(x)dx = 0$$

   (b) If the graph of f is symmetric with respect to the y-axis, then f is even. Prove for an even function f which is integrable on $[-a,a]$ that

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx \ .$$

   Interpret this result geometrically.

5. Prove Theorem 6-4c as a consequence of Lemmas 6-4a and 6-4b. Conversely, derive the Lemmas as corollaries of Theorem 6-4c.

6. Prove: If f and g are integrable where $g : x \longrightarrow |f(x)|$ on $[a,b]$, then

$$\left|\int_{a}^{b} f(x)dx\right| \leq \int_{a}^{b} |f(x)|dx \ .$$

7. Compute the values of the given integrals using Theorem 6-4c.

   (a) $\int_{2}^{3} (3x^2 - 5x + 1)dx$

   (b) $\int_{0}^{2} (x - 1)(x + 2)dx$

   (c) $\int_{-2}^{3} (x + 2)(x - 3)dx$

8. (a) Find the area of the region below the parabola $y = x^2 - 3$ above the x-axis and between the lines $x = -3$, $x = 3$.

   (b) Find the area of the region between the graph of $f : x \longrightarrow x^2 - x - 6$, the x-axis, and the lines $x = -2$, $x = 3$. First draw a rough sketch of f and indicate (by shading) the region whose area is to be computed.

9. Find all values of a for which

$$\int_{0}^{a} (x + x^2)dx = 0$$

10. Compute $\int_0^3 f(x)dx$ where

$$f(x) = \begin{cases} 2 - x^2 , & 0 \le x \le 1 \\ 5 - 4x , & 1 \le x \le 3 . \end{cases}$$

11. Verify that the following property holds for $f : x \longrightarrow x$

$$\int_a^b f(c - x)dx = \int_{c-b}^{c-a} f(x)dx .$$

Explain the property geometrically in terms of areas. Do you think that the property holds for other functions that are integrable? Justify your answer.

12. If a function f is periodic with period λ and integrable for all x, show that

$$\int_a^{a+n\lambda} f(x)dx = n\int_a^{a+\lambda} f(x)dx , \; (n, \text{integer}) .$$

Interpret geometrically.

13. Evaluate

$$\int_0^{100\pi} (1 + \sin 2x)dx$$

(assuming $\sin 2x$ is integrable).

14. Prove that if f is integrable on $[a,b]$ and if $f(x) \ge 0$ for all x in $[a,b]$, then

$$\int_a^b f(x)dx \ge 0 .$$

15. Interpret $\int_a^b f(x)dx$ in terms of area if $f(x)$ may take on both positive and negative values in $[a,b]$.

16. Interpret $\int_a^b \{g(x) - f(x)\}dx$ in terms of area if we admit the possibility that $f(x) > g(x)$ for some values of x in $[a,b]$.

17. Prove that if f and g are integrable over $[a,b]$, then

$$\left|\int_a^b \{g(x) - f(x)\}dx\right| \le \int_a^b |g(x)|dx + \int_a^b |f(x)|dx .$$

18. Let f and g be integrable and suppose that $f(x) \leq g(x)$ on $[a,b]$.

    (a) If the strong inequality $f(x) + \epsilon < g(x)$, for some $\epsilon > 0$, holds on any subinterval of $[a,b]$, prove the strong inequality

    $$\int_a^b f(x)dx < \int_a^b g(x)dx .$$

    (b) If f and g are continuous at $x = u$ in $[a,b]$ and $f(u) < g(u)$ prove that strong inequality holds as above.

19. If functions f and g are integrable, and $f(x) \leq h(x) \leq g(x)$ on $[a,b]$, does it follow that

    $$\int_a^b f(x)dx \leq \int_a^b h(x)dx \leq \int_a^b g(x)dx \ ?$$

    Illustrate by an example.

20. (a) Prove the Mean Value Theorem of integral calculus: If f is continuous and integrable on $[a,b]$, then there exists a value u in the open interval (a,b) such that

    $$\int_a^b f(x)dx = f(u)(b - a) .$$

    (b) Show that the value $f(u)$ in (a) satisfies

    $$f(u) = \lim_{h\sim 0} \frac{f_0 + f_1 + \dots + f_n}{n + 1}$$

    where $h = \frac{(b - a)}{n}$ and $f_k = f(a + kh)$ for $k = 0 , 1 , 2 , \dots ,$ n . Thus $f(u)$ can be interpreted as an extension of the idea of mean or arithmetic average to the values of a function on an interval.

21. If $\frac{a_0}{n + 1} + \frac{a_1}{n} + \dots + \frac{a_{n-1}}{2} + \frac{a_n}{1} = 0$, show that

    $$a_0x^n + a_1x^{n-1} + \dots + a_{n-1} + a_n = 0$$

    has at least one root in $(0,1)$.

22. (a) Prove that if $f(x)$ is integrable over $[a,b]$, then $|f(x)|$ is integrable over $[a,b]$.

    (b) Show that the converse is not true.

23. If f and g are integrable over $[a,b]$, then both $\max\{f,g\}$ and $\min\{f,g\}$ are also integrable over $[a,b]$.

24. (a) Let f and g be bounded and integrable over $[a,b]$.
    Prove (a) The function $f \cdot g$ is integrable over $[a,b]$;

    (b) If g is bounded away from zero, then $\frac{f}{g}$ is integrable on $[a,b]$.

25. If f and g are bounded and integrable, then $\int_a^b (\alpha f(x) + \beta g(x))^2 dx$ exists and is ≥ 0 for all constant α and β.

    Show from this that

$$\int_a^b f(x)^2 dx \cdot \int_a^b g(x)^2 dx \geq \left\{ \int_a^b f(x) \cdot g(x) dx \right\}^2 ,$$

    with equality if and only if (for f and g continuous)

$$f(x) = cg(x) \ , \ a \leq x \leq b \ .$$

    (Buniakowsky-Schwart Inequality. This is the integral analog of Cauchy's Inequality - Exercise A1-2, Number 16.)

26. If f is integrable and its graph is flexed upward on an interval $[0,a]$, show that

$$\int_0^a f(x) dx \geq af(\tfrac{a}{2}) \ .$$

    Interpret geometrically.

27. Show that

$$\sqrt{(a^2 + \tfrac{1}{3})(b^2 + \tfrac{1}{3})} \geq \int_0^1 \sqrt{(x^2 + a^2)(x^2 + b^2)}\, dx \ .$$

28. Show that

    (a) $\frac{1}{2} + \frac{3\sqrt{2}}{8} < \int_0^1 \sqrt{1 + x^3}\, dx < \frac{\sqrt{5}}{2}$.

    (b) $\frac{1}{2} + \frac{\sqrt{2}}{3} > \int_0^1 \frac{dx}{\sqrt{1 + x^3}} > \frac{2\sqrt{5}}{5}$.

29. Find a continuously differentiable function F (i.e., F' is continuous) in $[0,1]$ which satisfies the three conditions

(a) $F(0) = 0$, $F(1) = a$,

(b) $\int_0^1 F(x)^2 dx = \frac{a^2}{3}$, and

(c) $\int_0^1 F'(x)^2 dx$ is a minimum.

6-5. Further Applications of the Integral.

The interpretation of integral as area is but one of its many applications. In this section we shall give two other applications.

(i) Volumes of solids of revolution.

The general problem of evaluating the volume of a solid can be reduced to a succession of integrations. We shall not attack the general problem of volume, but shall solve the problem in terms of integration for a simple special case, that of a solid of revolution.

Let f be a nonnegative function on $[a,b]$, (Figure 6-5a). We define the solid of revolution generated by f on $[a,b]$ as the set of points swept over by the standard region under the graph of f in a complete rotation about the x-axis (Figure 6-5b).

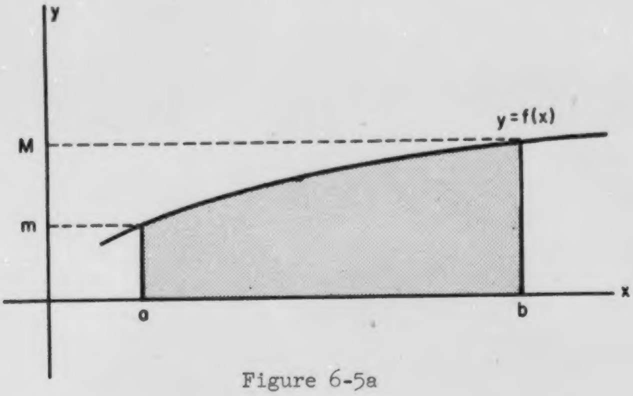

Figure 6-5a

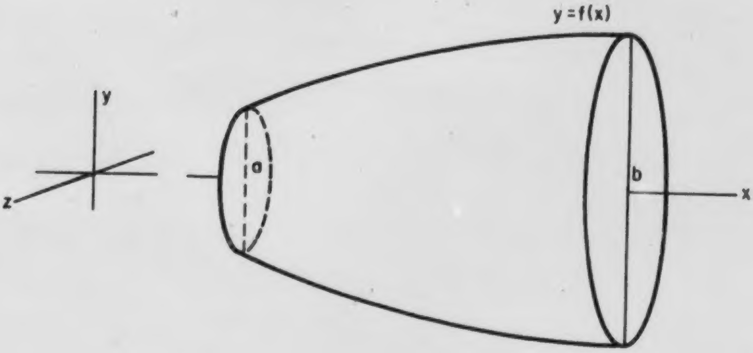

Figure 6-5b

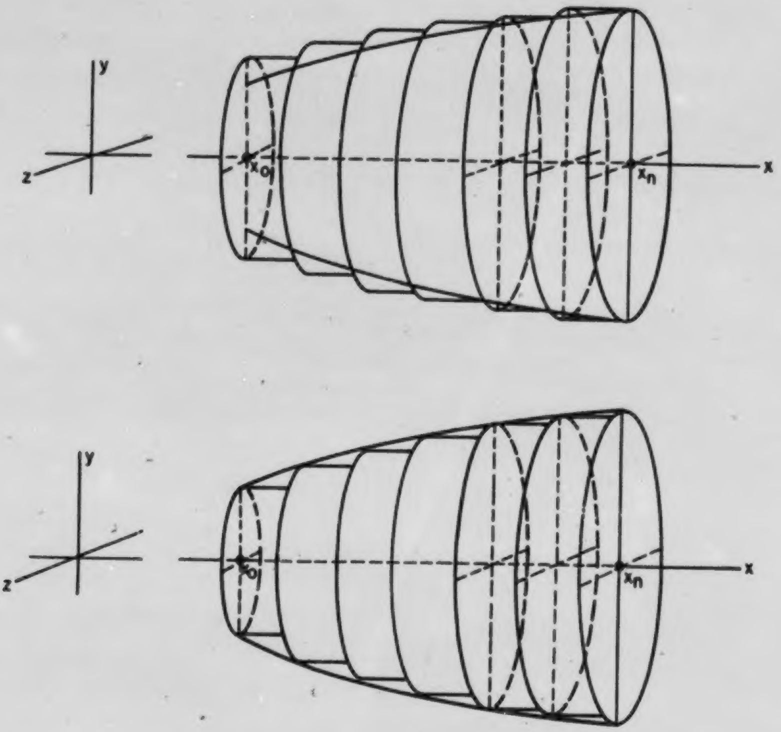

Figure 6-5c

If we choose a z-axis in a direction perpendicular to the x, y-plane this solid can be described as the set of points

$$\{(x,y,z) : a \le x \le b\ ,\ y^2 + z^2 \le [f(x)]^2\}\ .$$

We can easily obtain upper and lower estimates for the volume of this solid. In Figure 6-5a we have depicted f as a function which takes on the maximum value M and the minimum value m on $[a,b]$. The solid of revolution generated by f on $[a,b]$ is contained in an outer cylinder with a base of radius M and contains an inner cylinder with a base of radius m. Taking the formula for the volume of a right circular cylinder from elementary geometry, we have

$$\pi m^2(b - a) \le V \le \pi M^2(b - a)$$

where V is the volume of the solid.

We can divide up the solid by means of a partition of $[a,b]$ in a fashion similar to the subdivision of a standard planar region. The solid is cut into slices by parallel planes $x = x_k$ through the successive points of the partition. By obtaining upper and lower bounds for f in each interval of the subdivision of $[a,b]$ we can estimate the volume of each slice from above and below. Let M_k be an upper bound and m_k a lower bound for f on $[x_{k-1},x_k]$ A cylinder with a base of radius M_k and height $x_k - x_{k-1}$ contains the slice of the solid between the planes $x = x_{k-1}$ and $x = x_k$, and the slice in turn contains a cylinder with a base of radius m_k and height $x_k - x_{k-1}$. Adding the volumes of such cylinders we obtain estimates for the volume V in the form of upper and lower sums:

$$U = \sum_{k=1}^{n} \pi M_k^{\,2}(x_k - x_{k-1})$$

$$L = \sum_{k=1}^{n} \pi m_k^{\,2}(x_k - x_{k-1})$$

(Figure 6-5c). These are upper and lower sums for the function $g : x \longrightarrow \pi[f(x)]^2$. If g is integrable* we must have

$$V = \pi \int_a^b f(x)^2 \, dx .$$

Example 6-5a. We shall obtain the volume V of the segment of a sphere of radius r intercepted by a plane at distance a from the center (see Figure 6-5d). This is the solid of revolution generated by the circular arc $f(x) = \sqrt{r^2 - x^2}$ on the interval $[a,r]$. We have

$$V = \pi \int_a^r [f(x)]^2 dx$$

$$= \pi \int_a^r (r^2 - x^2) dx .$$

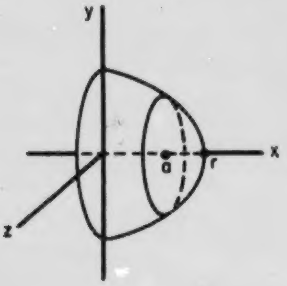

Figure 6-5d

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*In particular, if f is nonnegative and monotone, g will be monotone and therefore integrable.

Employing Theorem 6-4c and the special results of Examples 6-4a, b, we obtain

$$\begin{aligned} V &= \pi \int_a^r r^2 dx - \pi \int_a^r x^2 dx \\ &= \pi r^2(r - a) - \frac{\pi}{3}(r^3 - a^3) \\ &= \frac{\pi}{3}(r - a)(2r^2 - ar - a^2) \\ &= \frac{\pi}{3}(r - a)^2(2r + a) . \end{aligned}$$

(ii) Calculation of displacement from a known velocity function.

Let us consider the straight-line motion of a body for which we know the velocity v as a function of time, $v = f(t)$, and for which we wish to determine the position s of the body as a function of time, $s = \phi(t)$. Given the velocity of motion for a given time interval $[a,b]$ it should be possible to determine the total displacement or signed distance moved by the body in the given interval. Intuitively, if we divide the time interval into subintervals so small that the velocity does not change appreciably in each, then we can estimate the displacement in each subinterval. The sum of these estimates is an estimate of the total displacement.

Specifically, let

$$\sigma = \{t_0, t_1, t_2, \ldots, t_n\}$$

be a partition of $[a,b]$. If in the interval $[t_{k-1}, t_k]$ we have approximately $f(t) = v_k$ where v_k is constant, then the displacement for that time interval should be approximately $v_k(t_k - t_{k-1})$. If we take for v_k an upper or lower bound for the velocity on the interval $[t_{k-1}, t_k]$ we can form upper or lower sums and estimate the total displacement from above or below. In this way, we argue that the total displacement can be expressed as an integral

$$\int_a^b v \, dt \tag{1}$$

where v is given as a function of time, $v = f(t)$.

We have made out a plausible case for the expression of the total displacement as the integral (1), and we shall go on to prove it. Given the known function $f : t \longrightarrow v$, we wish to determine the total displacement $\phi(b) - \phi(a)$ where ϕ is the function which gives the position of the body at time t . Since f is the derivative of ϕ with respect to t

(Definition of velocity, Section 2-4), we have by the Law of the Mean

$$\phi(t_k) - \phi(t_{k-1}) = f(\tau_k)(t_k - t_{k-1})$$

where $t_{k-1} < \tau_k < t_k$. It follows that

$$\phi(b) - \phi(a) = \sum_{k=1}^{n} [\phi(t_k) - \phi(t_{k-1})]$$

$$= \sum_{k=1}^{n} f(\tau_k)(t_k - t_{k-1}) .$$

In this way the total displacement $\phi(b) - \phi(a)$ is expressed as a Riemann sum over any partition of $[a,b]$. If f is integrable over $[a,b]$ it follows (Theorem 6-3c) that

$$\phi(b) - \phi(a) = \int_a^b f(t)dt . \tag{2}$$

which is the result we sought to prove.

In actuality we have established (2) for any function ϕ which has an integrable derivative on $[a,b]$,

$$\phi(b) - \phi(a) = \int_a^b \phi'(t)dt . \tag{3}$$

This general result is the most important application of the concept of integral. In Chapter 7, we shall examine this and related results in detail.

Example 6-5b. As an immediate application of (3) we shall describe the motion of a body in free vertical fall near the surface of the earth. For this purpose we utilize the concept of acceleration, which is defined as the derivative of velocity with respect to time. We are given that the acceleration of a body in the gravitational field near the earth's surface is, for all practical purposes constant, about 32 ft./sec^2 . If z denotes the height of a freely falling body above the earth, we have for the velocity v ,

$$\frac{dz}{dt} = v , \tag{4}$$

and for the accelearation a ,

$$\frac{dv}{dt} = a . \tag{5}$$

Here we have taken the positive sense of displacement, velocity, and acceleration as upward; therefore we must set $a = -32$ in Equation (5). We take $t = 0$ as the time the motion is initiated and seek the velocity $v = f(\tau)$ and position $z = \phi(\tau)$ at each subsequent instant τ of the motion. From (3) we have (from Example 6-4b)

$$f(\tau) - f(0) = \int_0^\tau (-32)dt = -32\tau .$$

Setting $v_0 = f(0)$ above we obtain

$$v - v_0 = -32\tau$$

or

$$v = v_0 - 32\tau ,$$

where v_0 is the initial velocity; thus $f(t) = v_0 - 32t$. Entering this result in (4), we have

$$\frac{dz}{dt} = v_0 - 32t .$$

From (3) we conclude that

$$\phi(\tau) - \phi(0) = \int_0^\tau (v_0 - 32t)dt .$$

Employing the results of Examples 6-4b, c, and using Theorem 6-4c, we have

$$z - z_0 = v_0\tau - 32\frac{\tau^2}{2} ,$$

where z_0 is the initial position of the body. At time t after the initiation of the motion the position of the body is

$$(6) \qquad z = z_0 + v_0t - 16t^2 .$$

By successive differentiations we may check that Equations (4) and (5) are satisfied. We have not verified that (6) describes the only motion which is possible under the given initial conditions, but we shall see later that the initial conditions do uniquely determine the ensuing motion.

Exercises 6-5

1. Find the volume of the solid of revolution generated by $f : x \longrightarrow \sqrt{x}$ on $[0,1]$.

2. Use the procedure of this section to find the volume of a right circular cone of altitude h and base of radius r .

3. Obtain the formula for the volume of a sphere of radius r by first showing that the sphere is a solid of revolution.

4. Find the volume of the ellipsoid generated by rotating the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about its major axis. (Assume $a > b$.)

5. Find the volume of the segment of a sphere of radius r bounded by two parallel planes if the bases of the segment are at distances a and b from the center and are on the same side.

6. If the acceleration of a particle moving along a line is $3t - 2$ in centimeters per second per second at any time t in seconds and if the velocity is 2 centimeters per second when $t = 0$, find the distance covered during the first second.

7. A particle moves along a line so that its velocity at any time is given by $v = 4t^2 - 14t + 6$. Find the distance covered by the particle between the instants when it is at rest.

8. With what upward velocity must a ball be thrown if it is to reach a height of 100 feet above the point from which it is thrown?

9. A stone is thrown down from the top of a 200 foot tower with an initial velocity of 40 feet per second. How long will it take the stone to reach the ground? With what velocity will it hit the ground?

10. Answer the questions asked in Number 9 in the case where the stone is dropped from the tower.

11. If the stone is thrown straight up with a velocity of 40 feet per second from the top of the 200 foot tower, with what velocity will it strike the ground?

12. Find the volume of the solid obtained by revolving the region bounded by the parabola $y^2 = 4x$ and the line $y = x$ about the x-axis.

13. A cylindrical hole of radius 1 inch is drilled out along a diameter of a solid sphere of radius 4 inches. Find the volume of the material cut out.

14. Find the volume of the portion of a sphere remaining after a cylindrical hole is drilled out along its diameter if the length of the hole is H. Check your answer by considering some special cases.

Miscellaneous Exercises

1. If K is a constant prove that $\int_{\sin^2 t}^{\cos^2 t} K\,dx = K \cos 2t$.

2. Find the area of the region bounded by the graph of $f : x \rightarrow 6 + x - x^2$ and the line $y + x = 3$.

3. Over what intervals is the given function integrable?

   (a) $f : x \rightarrow \frac{1}{x}$

   (b) $g : x \rightarrow \frac{|x|}{x}$

4. For which of the functions of Number 3 is it possible to extend the domain so that the function g defined on the extended domain is integrable over every closed interval? Does the value of the integral of g over any interval which contains a point of the extension depend upon the way in which the domain of f is extended? In the light of your conclusion suggest a generalization of the definition of integral.

5. The area of the standard region of a function f over $[a,b]$ is given by

$$\frac{1}{6}(2a^3 - 2b^3 - 9a^2 + 9b^2) .$$

   Find a function f for which this is true. Is there more than one such function? Interpret your answer geometrically.

6. A point moves on a line such that after t seconds its velocity is $v = t^2 - 9t + 20$. If its position is $s = 0$ when $t = 0$, how far does the point move during the time when the velocity is nonpositive?

7. Show that

(a) $1 < \int_0^1 \frac{x^2 + a^2}{x^2 + 1}\,dx < a^2 , \quad (a > 1)$.

(b) $1 < \int_0^1 \frac{x^4 + x^2 + a^2}{x^4 + x^2 + 1}\,dx < a^2 , \quad (a > 1)$.

8. For a monotone function f on $[a,b]$ we have found that the upper and lower sums over a partition σ satisfy

$$U - L < \epsilon$$

provided the norm $\nu(\sigma)$ is sufficiently small,

$$\nu(\sigma) < \frac{\epsilon}{|f(b) - f(a)|}$$

when $f(b) \neq f(a)$. Since the integral I of f over $[a,b]$ lies between L and U we can estimate I by either sum within the tolerance ϵ . Show that the arithmetic mean $\frac{1}{2}[L + U]$ is an estimate of I within the finer tolerance $\frac{\epsilon}{2}$.

9. Find Riemann sums differing by less than 0.1 for the area of the standard region of $f : x \rightarrow \frac{-1}{x - 1}$ over $[-1,0]$.

10. If when one applies the brakes of an automobile a constant deceleration of c ft/sec^2 results, determine the value of c necessary to ensure that an automobile traveling at 40 mi./hr. will stop 50 feet from the place where the brakes are applied.

11. Show that

$$\int_a^b f(x)dx = \frac{b - a}{6}\left(f(a) + 4f\left(\frac{a + b}{2}\right) + f(b)\right)$$

where $f(x)$ is any quadratic function.

12. (a) Let f be continuous and increasing on $[a,b]$, g be its inverse, and set $\beta = f(a)$, $\alpha = f(b)$. Prove

$$\int_\alpha^\beta g(y)dy = \beta b - \alpha a - \int_a^b f(x)dx .$$

Interpret this result geometrically when a and α are positive. (Hint: Compare with Nos. 2-4 of Exercises 6-2.)

(b) Prove the formula and find a similar interpretation when f is decreasing on $[a,b]$.

13. Check the arc length formula by applying it to a segment of a straight line and comparing with the formula for the distance between two points on a straight line.

Chapter 7

BASIC INTEGRAL THEOREMS

7-1. Integrability.

Our purpose in this section is to settle the question of integrability for the functions which concern us most in this text. We have already shown (Theorem 6-3b) that a function f which is monotone on $[a,b]$ is integrable over $[a,b]$. In addition, we know that any linear combination of integrable functions is integrable (Theorem 6-4c). Coupling these results, we obtain a general class of integrable functions, the linear combinations of monotone functions. This class contains most functions studied in the calculus.

We generally require that a function be continuous and differentiable. Its derivative will usually have at most finitely many zeros in any closed interval (with the trivial exception of a constant function). Such a function has finitely many maxima and minima separating intervals in which the function is monotone. It is a member of the general class of piecewise monotone functions.

DEFINITION 7-1a. A function f is said to be piecewise or sectionally monotone on $[a,b]$ if there is a partition $\sigma = \{x_0, x_1, x_2, \ldots, x_n\}$ of $[a,b]$ such that the function f is monotone on each subinterval $[x_{k-1}, x_k]$.

The direct approach to the problem of showing that a piecewise monotone function is integrable would be to use the method of Theorem 6-3b of estimation from above and below for each interval where the function is monotone. That approach is left as an exercise. Instead we show that a piecewise monotone function can be described as a linear combination of monotone functions[*]. The idea is to decompose the representation of f as a piecewise monotone function into the sum of a weakly increasing function constructed from the ascending sections of the graph and a weakly decreasing function constructed from the descending sections. The method is illustrated in the following example.

Example 6-5. Consider the function $f : x \longrightarrow x^2$ on an interval $[a,b]$ where $a < 0 < b$, (Figure 7-1a). On the interval $[a,0]$ the function is decreasing and on the interval $[0,b]$, increasing. We can represent f as the sum of the two functions g and h where

$$g(x) = \begin{cases} 0 , & a \le x \le 0 , \\ x^2 , & 0 \le x \le b , \end{cases}$$

$$h(x) = \begin{cases} x^2 , & a \le x \le 0 , \\ 0 , & 0 \le x \le b , \end{cases}$$

and the function g is weakly increasing and h weakly decreasing on $[a,b]$. Since x^2 is the linear combination $f(x) = g(x) + h(x)$ of monotone functions we conclude that f is integrable on $[a,b]$.

---

[*] A linear combination of monotone functions may have bizarre properties. We show that a sum of two monotone functions need not be piecewise monotone. Consider the function

$$g : x \longrightarrow \begin{cases} 2x + x^2 \sin \frac{1}{x} & \text{for } 0 < x < \frac{1}{2} , \\ 0 & \text{for } x = 0 , \end{cases}$$

which has the derivative

$$g' : x \longrightarrow \begin{cases} 2 - \cos \frac{1}{x} + 2x \sin \frac{1}{x} & \text{for } 0 < x < \frac{1}{2} , \\ 2 & \text{for } x = 0 . \end{cases}$$

Thus $g'(x) > 0$ for $0 \le x < \frac{1}{2}$, and g is increasing. Certainly $h : x \longrightarrow -2x$ is decreasing, but the sum $f = g + h$ given by $f(x) = x^2 \sin \frac{1}{x}$ is not piecewise monotone on $[0,\frac{1}{2}]$.

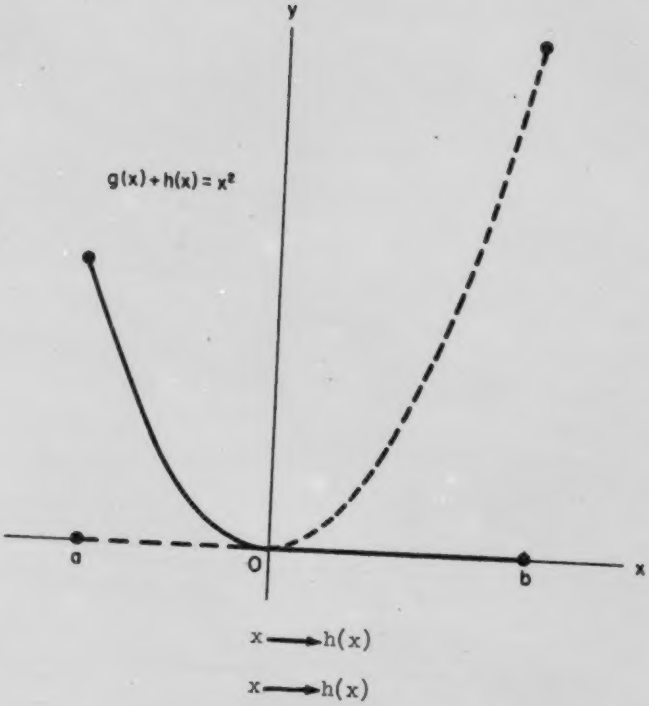

Figure 7-1a

In general, let f be monotone on each interval of the subdivision of $[a,b]$ defined by the partition $\sigma = \{x_0, x_1, x_2, \ldots, x_n\}$. We shall represent the function f in the form $f = g + h$ where g is weakly increasing and h is weakly decreasing. If f is weakly increasing on $[x_0, x_1]$, set $g(x) = f(x)$ and $h(x) = 0$ on $[x_0, x_1]$. If f is weakly decreasing set $g(x) = 0$ and $h(x) = f(x)$ on $[x_0, x_1]$. Now as x increases, if $f(x)$ increases (weakly) add the increment of $f(x)$ to $g(x)$ and keep $h(x)$ fixed; if $f(x)$ decreases keep $g(x)$ fixed and let $h(x)$ decrease by the same amount as $f(x)$. Thus, we proceed recursively: if f is weakly increasing in $[x_{k-1}, x_k]$ we define $g(x) = g(x_{k-1}) + (f(x) - f(x_{k-1}))$ and $h(x) = h(x_{k-1})$; otherwise we define g and h in $[x_{k-1}, x_k]$ by $g(x) = g(x_{k-1})$ and $h(x) = h(x_{k-1}) + (f(x) - f(x_{k-1}))$. A construction of this kind is depicted in Figure 7-1b.

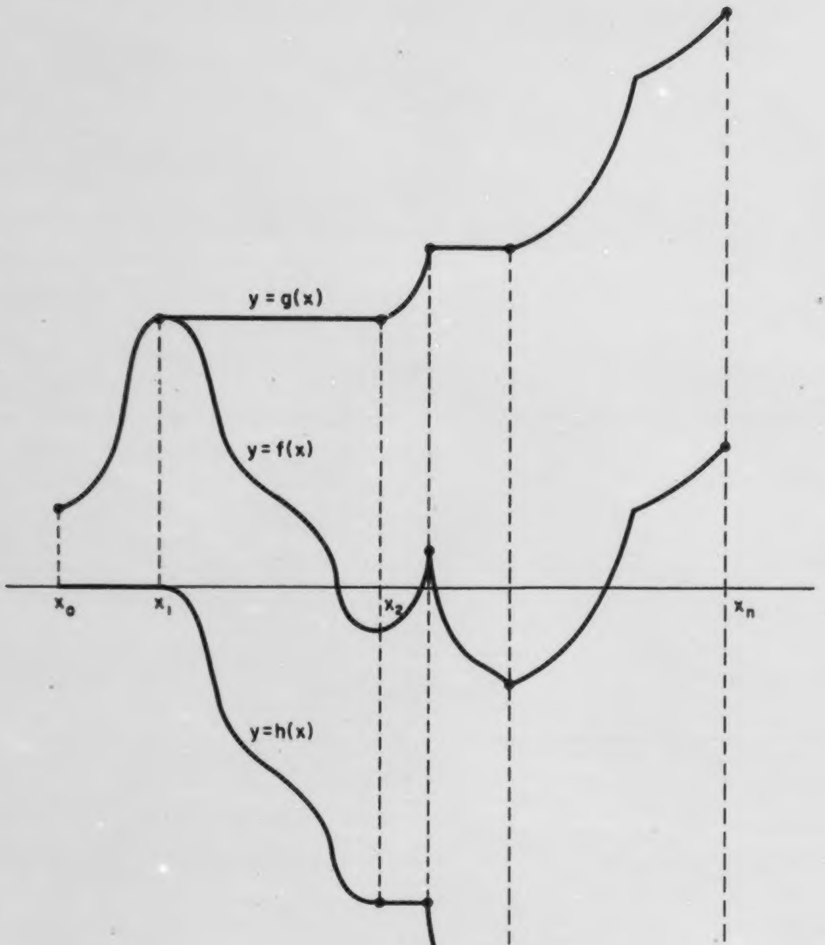
y = g(x)
y = f(x)
y = h(x)
x_0
x_1
x_2
x_n

Figure 7-1b

Lemma 7-1. A piecewise monotone function f can be represented in the form $g + h$ where g is weakly increasing and h is weakly decreasing.

Corollary. Any function which is piecewise monotone on a closed interval is integrable over the interval.

The last result is the basic integrability theorem employed in this text. For analysis and its applications, two classes of integrable functions have major significance: the piecewise monotone functions and the continuous functions. Most of the functions we are concerned with are in both classes, but a function can be in one class and not the other. For example, the useful function $x \longrightarrow \operatorname{sgn} x$ (Section A2-1) is monotone on $[-1,1]$ but not continuous; the Weierstrass function of Section A4-3 is continuous but has infinitely many strong local maxima and minima in every interval* and therefore cannot be monotone in any interval, no matter how small. We see, then, that a continuous function may not be piecewise monotone. Furthermore, it is not always possible to express a continuous function as a linear combination of monotone functions (Exercises A7-2, Nos. 1,2). The proof that every continuous function is integrable must then be a new venture. We do not have to prove integrability for all continuous functions to develop the calculus. Since it requires either the development of new concepts or a degree of analytical complexity, the proof is left to the appendix (Theorem A7-2).

Exercises 7-1

1. Show that any linear combination of monotone functions $\sum_{i=1}^{n} c_i f_i$ can be written as a sum of two functions $g + h$ where g is weakly increasing and h is weakly decreasing.

---

*A local extremum $f(u)$ is said to be strong if for all x in some deleted neighborhood of u we have $f(x) \neq f(u)$.

2. For each of the following express f as the sum of monotone functions g and h and give formulas for $g(x)$ and $h(x)$ in each of the subintervals where f is monotone.

   (a) $f : x \rightarrow \arcsin(\sin x)$, $-\pi \le x \le \pi$ (Exercises 4-5, No. 1(a))

   (b) $f : x \rightarrow \arcsin(\cos x)$, $-\pi \le x \le \pi$ (Exercises 4-5, No. 1(c))

   (c) $f : x \rightarrow 4x^5 + 5x^4 - 20x^3 - 50x^2 - 40x$ (Example 5-8a and Exercises 5-8, No. 1)

   (d) $f : x \rightarrow -x\sqrt{3 - x^2}$ (Exercises 5-8, No. 5)

   (e) $f : x \rightarrow x^{2/3}(x - 2)^2$ (Exercises 5-8, No. 7(b))

7-2. The Integral and its Derivative.

A function f which is integrable over $[a,b]$ is also integrable over any interval $[a,x]$ where $a < x < b$, (Lemma A6-2c). If f is integrable over $[a,b]$ we may then define a new function F on $[a,b]$ whose values are defined as integrals,

$$F : x \longrightarrow \int_a^x f(t)dt \; . \tag{1}$$

The formulas obtained in Examples 6-4a, b, c immediately yield the integrals

$$\int_a^x t^2 dt = \frac{x^3}{3} - \frac{a^3}{3} ,$$

$$\int_a^x C dt = Cx - Ca ,$$

$$\int_a^x t\, dt = \frac{x^2}{2} - \frac{a^2}{2} .$$

In each case, we observe that the derivative of the integral is the integrand (the function being integrated). In Section 6-5 we found other evidence of a reciprocal relation between differentiation and integration: if the function ϕ has the derivative f on $[a,b]$ and f is integrable over $[a,b]$, then $\phi(b) - \phi(a) = \int_a^b f(x)dx$. Is it true in general that the derivative of an integral is the integrand?

We pose the problem: to differentiate the integral F of f if the derivative exists and, incidentally, to find conditions under which the derivative exists. Thus, from (1) we wish to evaluate

$$F'(x) = \lim_{h \sim 0} \frac{F(x + h) - F(x)}{h} \; . \tag{2}$$

We have,

$$F(x + h) - F(x) = \int_a^{x+h} f(t)dt - \int_a^x f(t)dt \; , \tag{3}$$

and, for $h > 0$, if the two integrals are interpreted as areas, their difference is the area of the standard region under the graph of f on the interval $[x, x + h]$, (Figure 7-2). If f is continuous then for small h the values $f(t)$ will approximate $f(x)$ for t in $[x, x + h]$; the area of the standard region on $[x, x + h]$ may be expected to be close to that of the rectangle of height $f(x)$ on the same base. For a continuous function f we have the approximation

$$F(x + h) - F(x) \approx hf(x) .$$

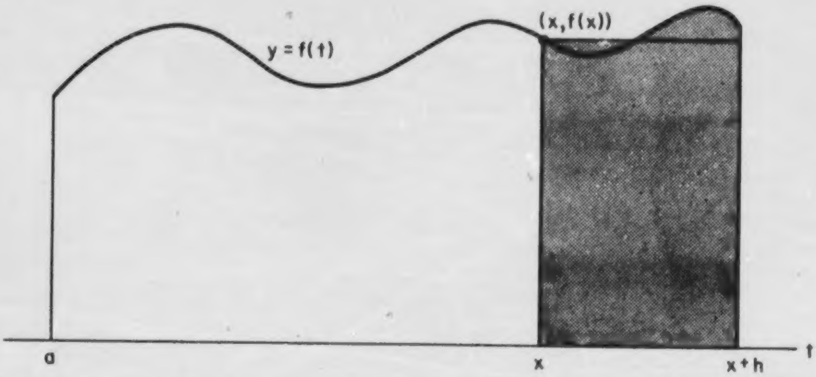

Figure 7-2

If we divide in (3) by h, and take the limit as h approaches 0, we anticipate that

$$F'(x) = \frac{d}{dx} \int_a^x f(t)dt = f(x) .$$

This result, which we have found intuitively, is now proved by a careful elaboration of these arguments.

Lemma 7-2. If f is integrable on an interval containing the points a and x, and continuous at x, then

$$\frac{d}{dx} \int_a^x f(t)dt = f(x) .$$

Proof. Let $F(x) = \int_a^x f(t)dt$, so that by Theorem 6-4b,

$$\frac{F(x + h) - F(x)}{h} = \frac{1}{h} \int_x^{x+h} f(t)dt .$$

From the continuity of f it follows that for any positive ϵ, we can ensure

$$f(x) - \epsilon < f(t) < f(x) + \epsilon \tag{4}$$

for all t in a sufficiently small neighborhood of x, say

$$|t - x| < \delta .$$

If we choose $0 < h < \delta$ the inequality (4) is satisfied for all values t in $[x, x + h]$. From Theorem 6-4a it follows that

$$\bigl(f(x) - \epsilon\bigr)h \leq \int_x^{x+h} f(t)dt \leq \bigl(f(x) + \epsilon\bigr)h . \tag{5}$$

Similarly, if h is negative, $0 > h > -\delta$, on employing Definition 6-4b, we obtain (5) with the inequalitites reversed. In either case, on division by h, we have

$$f(x) - \epsilon \leq \frac{1}{h} \int_x^{x+h} f(t)dt \leq f(x) + \epsilon \tag{6}$$

whence

$$\left| \frac{1}{h} \int_x^{x+h} f(t)dt - f(x) \right| \leq \epsilon .$$

We conclude that

$$F'(x) = \lim_{h \sim 0} \frac{F(x + h) - F(x)}{h} = \lim_{h \sim 0} \frac{1}{h} \int_x^{x+h} f(t)dt = f(x) .$$

Lemma 7-2 states that differentiation inverts the operation of integration. From a continuous function f we obtain a new function F by integration and the derivative of this new function is just the function f with which we started.

Exercises 7-2

1. Prove the inequality (6) for $h < 0$.
2. Prove

$$\frac{d}{dx} \int_x^a f(t)dt = -f(x) .$$

3. (a) Differentiate

$$\int_a^{g(x)} f(t)dt$$

under suitable restrictions on f and g. (Hint: Consider the integral as a composition of functions).

(b) Differentiate

$$\int_{h(x)}^{g(x)} f(t)dt$$

under suitable restrictions.

4. From the Law of the Mean and Lemma 7-2 derive the Mean Value Theorem (See Exercises 6-4, No. 20(a)); that is, prove if f is continuous on $[a,b]$, then

$$\int_a^b f(t)dt = f(u)(b - a)$$

for some u in the open interval (a,b).

7-3. The Fundamental Theorem.

If the effect of differentiation is to undo the work of integration it is natural to enquire about the performance of the two operations in the opposite order, whether integration reverses the operation of differentiation.

Suppose that F is differentiable and that F' is continuous[*] on an interval containing the points a and x. We wish to compare the integral of F' with F. From Lemma 7-2 we know that

$$\frac{d}{dx}\int_a^x F'(t)dt = F'(x) .$$

We have already proved (Corollary 2 to Theorem 5-4a) that if two functions have the same derivative they differ by a constant. It follows that

$$\int_a^x F'(t)dt = F(x) + C$$

where C is constant. Setting $x = a$ in this equation, we obtain

$$\int_a^a F'(t)dt = 0 = F(a) + C$$

whence

$$C = -F(a)$$

and

$$\int_a^x F'(t)dt = F(x) - F(a) .$$

Coupling this result with that of Lemma 7-2 we have

THEOREM 7-3. (The Fundamental Theorem of the Calculus). If f is continuous on an interval containing the points a and x, then

(1) $$\frac{d}{dx}\int_a^x f(t)dt = f(x) .$$

---

[*] If F' is continuous, then it is integrable (Theorem A7-2). Actually, as we have seen in Section 6-5, the result we now prove follows without assuming continuity solely from the integrability of F' on the ground that the integral is the limit of Riemann sums (Theorem A6-2).

Conversely, if F has a continuous derivative F' on an interval containing points a and x then

$$\int_a^x F'(t)dt = F(x) - F(a) . \tag{2}$$

In this remarkable result we have exhibited the intimate relation between derivative and integral. With this link the differential calculus and the integral calculus are seen not as two separate subjects but as aspects of a single discipline.

The great role of the Fundamental Theorem is to provide the means of transforming the formal calculus of derivatives into a formal calculus of integrals. Differentiation, as we have seen, involves much simpler analytical techniques than integration by summation. When the Fundamental Theorem can be applied to a problem of integration it represents a considerable economy of labor. For example, in Section 6-2 it was necessary to be ingenious in the art of summation to obtain the formula

$$\int_a^b x^2dx = \frac{b^3}{3} - \frac{a^3}{3} . \tag{3}$$

Without further investigation, we could not even be sure that the formula was valid for other than positive values a and b. Now we see at once from the differentiation formula

$$\frac{d}{dx}\left(\frac{x^3}{3}\right) = x^2$$

that Formula (3) is correct and that it holds for all values a and b. The formula

$$\int_0^a \cos x \, dx = \sin a ,$$

which requires even greater ingenuity to integrate by summation techniques, now follows directly from

$$D_x \sin x = \cos x .$$

We have devised no summation technique for obtaining the formula

$$\int_a^b x \sin x \, dx = \sin b - b \cos b - \sin a + a \cos a$$

but we can verify its correctness by calculating the derivative

$$\frac{d}{dx}(\sin x - x \cos x) = x \sin x .$$

According to the Fundamental Theorem any integral of f is a solution of the functional equation

$$DF = f . \tag{4}$$

A solution F of (4) is called an antiderivative[*] of f. An integral of f must be an antiderivative; an antiderivative of f may not be an integral, but can always be expressed as an integral plus a constant (Exercises 7-3, No. 4).

Since the distinction between antiderivative and integral is so slight we shall not attempt to preserve the distinction in later sections but refer to any solution F of (4) as an integral of f.

The class of all antiderivatives of f is denoted by an integration sign without ends of integration,

$$\int f(x)dx .$$

This class is called the indefinite integral of f. We put

$$\int f(x)dx = F(x) + C$$

where F is any particular antiderivative. This notation calls attention to the fact that the indefinite integral is a family of functions whose members are given by assigning values to the parameter C. Thus

$$\int x^2dx = \frac{x^3}{3} + C .$$

If the context does not make the distinction clear we shall sometimes refer to the integral $\int_a^b f(x)dx$ as a definite integral.

From any differentiation formula we may obtain a corresponding integration formula. In general, we have $\int f'(x)dx = f(x) + C$; for example, the formula

$$\frac{d}{dx} \sin x = \cos x$$

yields

$$\int \cos x \, dx = \sin x + C .$$

---

[*] An antiderivative F is called a primitive of f in many texts. The word, primitive, is opposed to "derivative", "primitive" denoting an original function from which the derivative is derived.

The formula

$$\lambda f(x) = \frac{d}{dx} \int \lambda f(x)dx$$

$$= \lambda \frac{d}{dx} \int f(x)dx$$

yields,

$$\int \lambda f(x)dx = \lambda \int f(x)dx \ . \tag{5}$$

One of the most important integration formulas corresponds to the chain rule of differentiation. Let $z = g(y)$ and $y = h(x)$ and set $f(x) = g\ h(x)$. From the differentiation formula

$$f'(x) = g'\ h(x)\ h'(x)$$

we obtain

$$\int g'\ h(x)\ h'(x)dx = f(x) + C \ .$$

In Leibnizian notation, the chain rule and the corresponding integral formula take on a simpler appearance:

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}$$

and

$$\int \frac{dz}{dx}\,dx = \int \frac{dz}{dy}\frac{dy}{dx}\,dx = z + C$$

In Chapter 10 we shall make extensive use of this result (Substitution Rule).

If we recognize the function f as the derivative of F on $[a,b]$ then we can obtain the integral of f over $[a,b]$ without appealing to summation techniques:

$$\int_a^b f(x)dx = F(b) - F(a)$$

We can immediately apply everything we know from Chapter 4 to integrate a broad class of functions. With this knowledge we have the power to calculate simply areas and volumes for an enormous variety of figures beyond the realm of elementary geometry.

Equation (4), $DF = f$, is a differential equation for F that is, a condition on the function given in the form of an equation which involves one or more derivatives of F . Often in applications the most convenient formulation of a problem is one given in terms of a differential equation. Perhaps the most significant application of the concept of integral is its interpretation as a solution of a differential equation.

Exercises 7-3

1. For each of the following differentiation formulas write the corresponding integration formula:

   (a) $\frac{d}{dx}(x^3 - 3x) = 3x^2 - 3$,

   (b) $\frac{d}{dx}\left(\frac{1}{1 + x^2}\right) = -\frac{2x}{(1 + x^2)^2}$,

   (c) $\frac{d\sqrt{x}}{dx} = \frac{1}{2\sqrt{x}}$,

   (d) $\frac{d(\sin^2 x)}{dx} = \sin 2x$.

2. Find the given derivative and then write the corresponding integration formula:

   (a) $\frac{d}{dx} a \cos bx$,

   (b) $\frac{d}{dx} \tan x^2$,

   (c) $\frac{d}{dx} \arctan \sqrt{x}$,

   (d) $\frac{d}{dx} \sqrt{1 - \cos x}$.

3. Verify the following integration formulas.

   (a) $\int \sec^2 ax \, dx = \frac{1}{a} \tan ax + C$.

   (b) $\int (a + bx)^r dx = \frac{(a + bx)^{r+1}}{b(r + 1)} + C$, $r \neq -1$.

   (c) $\int \frac{ax}{\sqrt{ax^2 + b}} dx = \sqrt{ax^2 + b} + C$.

   (d) $2\int \sin^2 x \sin 2x \, dx = \sin^4 x + C$.

   (e) $\int \frac{1}{\sqrt{x - x^2}} dx = 2 \arcsin \sqrt{x} + C$.

4. (a) We have

$$\frac{d}{dx}\frac{1}{1+x^2} = -\frac{2x}{(1+x^2)^2} ,$$

hence $\frac{1}{1+x^2}$ is an antiderivative of $-\frac{2x}{(1+x^2)^2}$.

Prove that $\frac{1}{1+x^2}$, although it is an antiderivative of $-\frac{2x}{(1+x^2)^2}$, is not an integral $\int_a^x -\frac{2x}{(1+x^2)^2}\,dx$.

(b) Give a necessary and sufficient condition that an antiderivative F of a function f continuous on the domain $[a,b]$ be an integral.

5. The initial value problem for the differential equation

$$DF = f$$

is to determine the function F when f is given and an "initial value" $F(a)$ is specified. Show under suitable conditions that there exists just one function F satisfying these conditions.

6. Do No. 5 for the differential equation

$$xF'(x) + F(x) = f(x) .$$

7. If f has an integrable derivative on $[a,b]$, prove that f can be represented as the sum of two monotone functions, $f = g + h$. (Hint: Consider the integrals of $f'(t) \pm |f'(t)|$.)

8. Use the Fundamental Theorem to derive the linearity of integration (Theorem 6-4c) from the linearity of differentiation (Theorem 4-2a).

9. A horizontal translation of a graph in the plane $(x,y) \longrightarrow (x + c , y)$ amounts only to a lateral shift for the standard region under the graph (Figure 7-3).

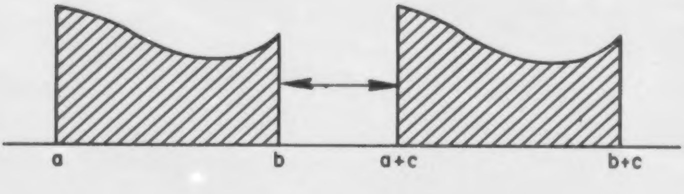

Figure 7-3

Verify that the expression of this geometrical property in terms of integrals is given by the formula

$$\int_a^b f(x)dx = \int_{a+c}^{b+c} f(x - c)dx$$

(a) Derive the preceding formula using only the methods of Sections 6-1 to 6-4.

(b) Derive the formula using the Fundamental Theorem.

10. Equation (5) can be interpreted geometrically as stating that a uniform change of vertical scale by the factor λ, that is, a transformation $(x,y) \longrightarrow (x,\lambda y)$, multiplies areas by λ. Since there are no preferred directions in the plane--the choice of coordinate axes is only a useful convention--the same result must be true of a uniform change of scale in any direction. In particular, a scale transformation $(x,y) \longrightarrow (\lambda x,y)$ in the x direction must multiply areas by the factor λ. Verify that the expression of this geometrical property in terms of integrals is given by the formula

$$\int_a^b f(x)dx = \int_{a\lambda}^{b\lambda} \frac{1}{\lambda} f(\frac{x}{\lambda})dx , \qquad \lambda \neq 0 .$$

(a) Derive this formula using only the methods of Sections 6-1 to 6-4.

(b) Derive this formula using the Fundamental Theorem.

(c) The number π is defined as the area of a circle of radius 1. Prove from the results of this exercise alone that the area of a circle of radius r is πr^2.

(d) By the method of (c) find the area of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 .$$

11. Prove that

$$\int_a^b f(x)dx = \int_{(a+c)\lambda}^{(b+c)\lambda} \frac{1}{\lambda} f(\frac{y}{\lambda} - c)dy , \qquad \lambda \neq 0 .$$

12. Using No. 11, show that

(a) $\int_0^{\pi} xf(\sin x)dx = \frac{\pi}{2} \int_0^{\pi} f(\sin x)dx$

(assuming the integrals exist).

(b) Evaluate $\int_0^{\pi} x \sin x \, dx$.

7-4. Formal Integration.

The variety of functions which we know how to differentiate is already enormous. Given a function whose integral we wish to evaluate there is a good chance that it is expressible as the derivative of some known function. If so, the integral can be evaluated easily. The approach to the problem of evaluating integrals through the Fundamental Theorem is subtle and indirect, but the gain in formal simplicity is remarkable.

It is convenient when $F(x)$ is given by a long expression to use the abbreviated notation

$$F(x)\Big|_a^b = F(b) - F(a) .$$

for $\int_a^b F'(t)dt$. We shall generally use this notation for specific numerical integrations.

Some integrals can be brought into the form of the derivative of a known function with slight algebraic manipulations. For example, $4x^5$ apart from a constant factor is the derivative of x^6 . We have, then,

$$\int_a^b 4x^5 dx = \frac{2}{3}\int_a^b 6x^5 dx = \frac{2}{3}x^6\Big|_a^b = \frac{2}{3}(b^6 - a^6) .$$

Again, apart from a constant factor, $\cos 2x$ is recognized as the derivative of $\sin 2x$; thus,

$$\int_a^b \cos 2x\, dx = \frac{1}{2}\int_a^b 2\cos 2x\, dx = \frac{1}{2}\sin 2x\Big|_a^b = \frac{\sin 2b - \sin 2a}{2} .$$

Such manipulations with constant factors are so easy that we shall usually employ them without comment from now on.

A more interesting example is

$$\int_0^{1/2} (1 - 2x)^9\, dx .$$

We may evaluate this integral by taking the binomial expansion of the integrand and integrating term-by-term. The more insightful approach is to recognize that, apart from a constant factor, the integrand is the derivative of $(1 - 2x)^{10}$:

$$D_x(1 - 2x)^{10} = 10(1 - 2x)^9(-2) .$$

We then have

$$\begin{aligned}\int_0^{1/2} (1 - 2x)^9 dx &= -\frac{1}{20}(1 - 2x)^{10}\Big|_0^{1/2} \\ &= -\frac{1}{20}(0 - 1) \\ &= \frac{1}{20} .\end{aligned}$$

Other general techniques of formal integration will be treated in Chapter 10.

In applying the formulas to evaluate definite integrals we must be careful to be sure that the integrand f is continuous in some interval containing the ends of integration. Otherwise the integration is meaningless. In evaluating $\int x^r dx$, for example, if the rational number r is negative then $x \rightarrow x^r$ is not continuous at $x = 0$. The integration

$$\int_a^b x^r dx = \frac{b^{r+1} - a^{r+1}}{r + 1}$$

for negative r will be meaningful in general only if a and b are positive The same caveat applies to $\int \sec^2 x \, dx$ and $\int \frac{1}{\sqrt{1 - x^2}} dx$. In the first the ends of integration must lie between consecutive zeros of $\cos x$ and, in the second, between 1 and -1.

The Fundamental Theorem is an extremely powerful tool for solving problems of integration, but it should not be thought that the Fundamental Theorem is the answer to all problems of integration. Consider

$$\int_a^b \frac{1}{x} dx ,$$

(where a and b lie in an interval where $\frac{1}{x}$ is defined; i.e., $ab > 0$). Since $\frac{1}{x}$ is monotone, the integral exists, but the problem of formal integration is another question. We have found no function F for which $F'(x) = \frac{1}{x}$. Furthermore, although the reasons for this are not evident, we could not find such a function by differentiating any of the functions treated in Chapter 4, including all functions which can be formed from them by rational combination, inversion, and composition. The integral $F(x) = \int_1^x \frac{1}{t} dt$ is an important function and we shall investigate its properties in Chapter 8.

Exercises 7-4

1. Use knowledge of derivatives previously obtained to find an expression for each of the following.

   (a) $\int \sin(ax + b)\,dx$

   (b) $\int \sqrt{ax + b}\,dx$

   (c) $\int (5 + x)^{10}\,dx$

   (d) $\int \frac{1}{(a + x)^{2/3}}\,dx$

   (e) $\int \frac{1}{\cos^2(ax)}\,dx$

   (f) $\int 2x \sin x^2\,dx$

   (g) $\int x^2(1 + 3x^3)^{2/3}\,dx$

   (h) $\int \frac{1}{\sqrt{1 - 4x^2}}\,dx$

   (i) $\int \tan^2 bx\,dx$

   (j) $\int \frac{x}{\sqrt{1 - x^2}}\,dx$

   (k) $\int \frac{\sqrt{1 - x}}{\sqrt{1 + x}}\,dx$

   (ℓ) $\int \frac{2x\,dx}{1 + x^4}$

   (m) $\int \frac{x^2\,dx}{\sqrt{1 - x^6}}$

2. Evaluate each of the following integrals.

   (a) $\int_0^{\pi} |\cos t|\,dt$

   (b) $\int_0^{1/2} \frac{1}{1 + 4x^2}\,dx$

   (c) $\int_{-\frac{\pi}{4}}^{0} \left(\frac{2}{\cos t}\right)^2 dt$

   (d) $\int_0^{\pi} D_x \sin^2 x\,dx$

   (e) $\int_0^{\pi} x \sin^2 nx\,dx$, n an integer

   (f) $\int_0^{\pi} x \cos^4 x\,dx$

3. Evaluate the following integrals:

   (a) $\int_{-1}^{1} x^{2/3}\,dx$

   (b) $\int_0^{2} (x - 1)^{2/3}\,dx$

   (c) $\int_a^{b} \frac{1}{\sqrt{1 - x^2}}\,dx$

   What are the restrictions on a and b ?

(d) $\int_a^b \tan x \sec x \, dx$

What are the restrictions on a and b ?

(e) $\int_1^2 (x^2 + 1)^{1/2} x \, dx$.

(f) $\int_a^b h'(gf(x)) \, g'(f(x)) \, f'(x) dx$.

(g) $\int_a^b \sin x^3 \cos x^3 \, x^2 \, dx$.

4. Consider the statement

$$\int_0^1 (1 - t^2)^{-1/2} \, dt = \frac{\pi}{2} .$$

By obtaining the corresponding indefinite integral, show how we might be led to believe this statement. Explain why the statement has no meaning in the present context and show how it can be given a meaning.

5. Find the lengths of the following curves between the indicates points.

(a) $y = \frac{2}{3}(x - 1)^{3/2}$, $(1,0)$ to $(10,18)$.

(b) $y = x^3 + \frac{1}{12x}$, $(1,\frac{13}{12})$ to $(2,\frac{193}{24})$.

(c) $y = \frac{x^4}{4} + \frac{x^{-2}}{8}$, from $x = a$ to $x = b$, where $0 < a < b$.

7-5. Estimates of Integrals.

When faced with a function f for which a formal integral can be obtained only with difficulty, if at all, we may yet obtain a practical estimation of the integral of f by integrating approximations to f for which integration is simple. If we can estimate f from above and below then we may apply Theorem 6-4a to obtain upper and lower bounds for the integral.

Example 7-5a. The inequality

$$1 - x^2 \leq \frac{1}{1 + x^2} \leq 1 , \tag{1}$$

is a direct consequence of the relation

$$(1 - x^4) = (1 - x^2)(1 + x^2) \leq 1 \leq 1 + x^2 .$$

From (1), we conclude that

$$\int_0^t (1 - x^2)dx \leq \int_0^t \frac{dx}{1 + x^2} \leq \int_0^t dx ,$$

and, $D \arctan x = \frac{1}{1+x^2}$, (Section 4-5),

$$t - \frac{t^3}{3} \leq \arctan t \leq t .$$

We approximate $\arctan t$ by the value $t - \frac{t^3}{3}$. The error can be no greater than $\frac{t^3}{3}$. For small values of t we can therefore obtain an excellent approximation to $\arctan t$.

Such an estimate can be used to estimate π. For example, we have

$$\arctan \frac{1}{\sqrt{3}} = \frac{\pi}{6} .$$

Consequently

$$\frac{\pi}{6} = \frac{1}{\sqrt{3}} - \frac{1}{3}\left(\frac{1}{\sqrt{3}}\right)^{-3} + e$$

where, for the error e, we have $|e| \leq \frac{1}{3}\left(\frac{1}{\sqrt{3}}\right)^3$. We have $\sqrt{3} = 1.732$, approximately. The maximum possible error in the estimate of π is given approximately by

$$6\left[\frac{1}{3}\left(\frac{1}{\sqrt{3}}\right)^3\right] = \frac{2}{3}\left(\frac{1}{\sqrt{3}}\right) = \frac{2\sqrt{3}}{9} = \frac{2}{9} \times 1.7 = 0.4 .$$

For π we have approximately

$$\pi = \frac{6}{\sqrt{3}}\left[1 - \frac{1}{3}\left(\frac{1}{\sqrt{3}}\right)^2\right] = \frac{6\sqrt{3}}{3} \cdot \frac{8}{9}$$

$$\approx (\frac{16}{9})(1.73) \approx 3.08$$

Accepting $\pi = 3.14 \ldots$, we see that the actual error is considerably smaller than the estimated maximum error.

Example 7-5b. From the inequality

$$0 \leq \cos x \leq 1 \quad \text{for} \quad 0 \leq x \leq \frac{\pi}{2}$$

we obtain by Theorem 6-4a

$$\int_0^t 0\,dx \leq \int_0^t \cos x\,dx \leq \int_0^t 1\,dx$$

whence, by $D \sin x = \cos x$

$$0 \leq \sin t \leq t , \qquad (0 \leq t \leq \frac{\pi}{2})$$

Integrating now from 0 to x, we obtain for x in $[0,\frac{\pi}{2}]$

$$\int_0^x 0\,dt \leq \int_0^x \sin t\,dt \leq \int_0^x t\,dt$$

or

$$0 \leq -\cos x + 1 \leq \frac{x^2}{2} ,$$

whence

$$1 - \frac{x^2}{2} \leq \cos x \leq 1 , \qquad (0 \leq x \leq \frac{\pi}{2}) .$$

On integrating again from 0 to t we obtain

$$t - \frac{t^3}{6} \leq \sin t \leq t , \qquad (0 \leq t \leq \frac{\pi}{2}) .$$

(Compare the result of Exercises 5-7, No. 9b). A further integration yields

$$\frac{x^2}{2} - \frac{x^4}{24} \leq -\cos x + 1 \leq \frac{x^2}{2} ,$$

whence,

$$1 - \frac{x^2}{2} \leq \cos x \leq 1 - \frac{x^2}{2} + \frac{x^4}{24} , \qquad (0 \leq x \leq \frac{\pi}{2}) .$$

Integrating once more, we have

$$t - \frac{t^3}{6} \leq \sin t \leq t - \frac{t^3}{6} + \frac{t^5}{120} , \qquad (0 \leq t \leq \frac{\pi}{2}) .$$

Continuing in this way, we can get bounds for $\cos x$ and $\sin t$, where at each stage we add terms of successively higher degree. It is possible to show for a fixed value of x or t that these estimates can be made to approximate $\cos x$ and $\sin t$ within any given positive tolerance ϵ simply by continuing the process far enough. However, the estimates are especially useful for the approximate numerical calculation of the sine and cosine when x and t are smaller than 1 . The difference between the upper and lower estimates is proportional to a high power of a positive number which is less than 1 and the error is accordingly small. For example, from the last inequality, we have approximately

$$\sin 0.1 = 0.0998333$$

with an error of at most one unit in the last place. Values of the trigonometric functions for larger x and t can be calculated in terms of the values for lesser x and t by use of trigonometric identities, e.g.,

$$\sin 0.2 = 2(\sin 0.1)(\cos 0.1) .$$

The tables of the trigonometric functions are computed by methods similar to these.

In higher analysis and applications, approximations and estimates such as we have exhibited here are often far more important than explicit representations even when they are obtainable. For the examples given here we have, in the first instance, a familiar function, arctan, but no simple way to calculate its values. We represent the function as an integral of a rational function, approximate the rational function by polynomials and integrate to obtain simple polynomial approximations to arctan. The approximations are far more convenient for numerical purposes than the explicit name, arctan.

In the second instance we push the concept of estimating an integral from estimates of the integrand in a very favorable circumstance. We use the cycling of the sine and cosine functions under repeated integration to improve our initial estimates. (This is done again in Chapter 8 for the exponential function.) However, the basic idea in general is not that of an integration cycle, but the use of a bound on a higher order derivative to obtain estimates for a function; this idea will be exploited in the proof of Taylor's Theorem (Chapter 13).

Exercises 7-5

1. (a) Obtain good approximations to $\sin \frac{1}{5}$ and $\cos \frac{1}{10}$ and give a tolerance within which you are sure of the accuracy of these values.

   (b) Provide a general formula for these estimates and prove by mathematical induction that the general formula is correct.

2. Evaluate:

   (a) $\lim\limits_{x \to 0} \dfrac{6 \sin x - 6x + x^3}{x^5}$.

   (b) $\lim\limits_{x \to 0} \dfrac{\cos 2x - 1 + 2x^2}{x^4}$.

   (c) $\lim\limits_{x \to 0} \dfrac{\sin x - x}{x(1 - \cos x)}$.

3. (a) Prove

$$1 - x^2 \leq \frac{1}{1 + x^2} \leq 1 - x^2 + x^4 .$$

   (b) Prove, in general, that

$$1 - x^2 + x^4 \dots + (-x^2)^{2p-1} \leq \frac{1}{1 + x^2} \leq 1 - x^2 + x^4 + \dots + (-x^2)^{2q}$$

   where p and q are any natural numbers.

   (c) Use the results of (a) and (b) to obtain upper and lower estimates for $\arctan x$.

   (d) Obtain a better approximation to π than that of Example 7-5a and show that the approximation is accurate within a closer tolerance.

Miscellaneous Exercises

1. Determine a number k for which each of the following integration formulas is correct.

   (a) $\int (5x + 2)^7 \, dx = k(5x + 2)^8 + C.$

   (b) $\int k \sqrt{3x - 2} \, dx = (3x - 2)^{3/2} + C.$

   (c) $\int \sin ax \, dx = k \cos ax + C.$

   (d) $\int \cos 2x \, dx = k \sin x \cos x + C.$

2. Compute each of the following integrals

   (a) $\int_0^{\pi/4} \sec^2 x \, dx$.

   (b) $\int_0^{\pi/2} \sin x \cos x \, dx$

   (c) $\int_0^{3\pi/2} |\sin x| dx$

   (d) $\int_0^{\pi/2} \sqrt{1 + \sin 2x} \, dx$

   (e) $\int_0^{\pi/2} \cos^4 x \, dx$.

   (f) $\int_0^{\pi} \sin^{100} x \cos^{99} x \, dx$

   Hint: See Exercises 6-4, No. 4.

   (g) $\int_0^{\pi/4} \sin^4 x \cos^3 x \, dx$.

   (h) $\int_0^{\pi/2} x \sin x \cos x \, dx$.

   Hint: Use Exercises 7-3, No. 12.

3. Find the area of the given standard region.

   (a) $f : x \longrightarrow \sqrt{2x + 1}$ over $[0,2]$

   (b) $f : x \longrightarrow \dfrac{1}{\sqrt{2x + 1}}$ over $[0,2]$

   (c) $f : x \longrightarrow x \sqrt{2x^2 + 1}$ over $[0,2]$

   (d) $f : x \longrightarrow \dfrac{x}{\sqrt{2x + 1}}$ over $[0,2]$

4. Compute each of the following integrals.

   (a) $\int_0^{\pi/2} (\sin x - 2 \cos x)dx$

   (b) $\int_{1/2}^{1} (2x + \csc^2 x)dx$

   (c) $\int_0^{\pi/2} \sin(x + n\pi)dx$, (n an integer)

   (d) $\int_0^{\pi} \sqrt{\frac{1 + \cos 2x}{2}}\, dx$

   (e) $\int_0^{100\pi} (2 \sin x - \sin 2x)^{99}\, dx$

5. Find the area of the standard region $f : x \to \sqrt{\frac{1 + \cos 2x}{2}}$ over $[0,2\pi]$.

6. Solve each of the following differential equations subject to the prescribed conditions.

   (a) $\frac{dy}{dx} = x\sqrt{1 + x^2}$, $x = 0$, $y = -3$

   (b) $\frac{dv}{dt} = \sqrt[3]{2t + 1}$, $t = 0$, $v = v_0$

   (c) $\frac{ds}{dt} = \frac{3}{8}(2t + 1)^{4/3} + v_0 - \frac{3}{8}$, $t = 0$, $s = s_0$.

7. Solve the differential equation

$$t^2 \frac{dy}{dt} + 2ty = \cos \pi t$$

   subject to the condition $t = 1$, $y = 0$.

8. The standard region of $f : x \to \frac{1}{x}$ over $[2,5]$ is rotated about the y-axis. Find the volume of the solid generated.

9. Find the area of the region bounded by the parabola $x = (y - 1)^2$ and the line $y - x + 1 = 0$.

10. Find the area of the region bounded by the graph of $f : x \to -x\sqrt{3 - x^2}$ and the x-axis. (See Exercises 5-8, No. 5.)

11. Find the area of the region bounded by the curves $y = x^5$ and $y = x^3$.

12. The region bounded by the curves $y = x^5$ and $y = x^3$ is rotated about the x-axis. Find the volume of the solid generated. Find the volume of the solid generated by rotating the region about the y-axis.

13. What is the mean value of the function $f : x \to \sin x$ over the interval $[0, \frac{\pi}{2}]$? Of $g : x \to \sin^2 x$ over $[0, \frac{\pi}{2}]$? (See Exercises 6-4, No. 20b.)

14. Find the area bounded by the graphs of $f : x \to \cos^2 x - \sin 2x + 1$ and $g : x \to \cos 2x - \sin^2 x + 1$ between the lines $x = \frac{\pi}{4}$ and $x = \frac{3\pi}{4}$.

15. (a) Find the volume of the paraboloid generated by rotating the region in the first quadrant bounded by the parabola $y = x^2$, the y-axis, and the line $y = h$, $h > 0$.

    (b) Compare the volume of the paraboloid with the volume of a right circular cone of equal base-radius and height.

16. Find the volume of a goblet in the form of a paraboloid of revolution which is 4 inches in diameter and 3 inches deep.

17. Starting with

$$\frac{\pi}{4} = 7{,}854 \arctan \frac{1}{10{,}000} - \arctan \frac{1}{545{,}261}$$

(D.H. Lehmer), use Number 3(c) of Exercises 7-5 to find a rational approximation to π correct to at least 15 decimal places.

18. Evaluate:

(a) $$\lim_{n\to\infty} \left\{ \frac{1}{\sqrt{n^2 + n}} + \frac{1}{\sqrt{n^2 + 2n}} + \cdots + \frac{1}{\sqrt{n^2 + n \cdot n}} \right\}.$$

(b) $$\lim_{n\to\infty} \sum_{r=0}^{n} \frac{1}{\sqrt{n^2 + r\sqrt{n}}} .$$

(c) $$\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{2n} \frac{r}{\sqrt{n^2 + r^2}} .$$

(d) $$\lim_{n\to\infty} \left\{ \frac{n}{n^2 + 1^2} + \frac{n}{n^2 + 2^2} + \cdots + \frac{n}{n^2 + n^2} \right\} .$$

Chapter 8

LOGARITHMIC AND EXPONENTIAL FUNCTIONS

8-1. Introduction.

In this chapter we shall exhibit the strength of the integral theorems of Chapters 6 and 7 by using them to frame precise definitions of power, exponential and logarithmic functions and to derive the properties of these functions in simple but logical fashion. First we proceed intuitively from the familiar algebraic properties of powers, properties which have yet to be conclusively proved in all generality. Beginning with the intuitive concept of power, we consider functions based upon that concept: powers, exponentials, and logarithms. When we attempt to differentiate these functions we shall see that derivatives of logarithms are especially simple. It is tempting then to use the fundamental theorem and treat logarithms as antiderivatives rather than define them in terms of their algebraic properties. We do so, and from the definition of the logarithm as an integral obtain the properties of logarithmic, exponential and power functions simply, naturally, and convincingly.

First we explore the properties of exponentials and logarithms based on an intuitive approach to the idea of power. As we proceed, difficulties arise in defining these functions in complete generality, but we shall be content to leave these difficulties unresolved for a while. When we obtain the definition of logarithm as an integral we shall return to these problems. With the resources of the integral theorems of Chapters 6 and 7 the resolution of these difficulties will turn out to be quite simple.

The original meaning of a^n, the n-th power of a, was confined to positive integral values of n: for each natural number n, a^n is defined as the product of n factors

$$a^n = a \cdot a \cdot a \cdot \ldots \cdot a .$$

In particular a^1 is the "product" of one factor, $a^1 = a$. In the symbol a^n, a is called the base and n the exponent. For positive integral exponents m and n we have the general laws

(1) $$a^m \cdot a^n = a^{m+n}$$

(2) $$(a^m)^n = a^{mn}$$

(3) $$(ab)^n = a^n b^n \; .$$

The observation that

$$\frac{a^n}{a^m} = a^{n-m} \qquad (n > m \; , \; a \neq 0)$$

leads to the definition of powers with zero and negative exponents simply by an extension of this formula to the cases where $n = m$ and $n < m$:

$$a^0 = 1 \qquad (a \neq 0)$$

and

$$a^{-n} = \frac{1}{a^n} \qquad (a \neq 0) \; .$$

If we require $a \neq 0$ and $b \neq 0$ then the laws (1), (2), (3) for positive exponents become valid for all integral exponents.

We may think of the introduction of negative exponents as an extension of the range of validity of rule (1) to include the additive identity and inverses; that is, we define a^{-n} and a^0 so that

$$a^{-n} \cdot a^n = a^{-n+n} = a^0 \; .$$

In exactly the same spirit we are led to fractional exponents if we extend rule (2) by introducing the multiplicative inverses; that is, we define $a^{1/n}$ so that

$$(a^{1/n})^n = a^1 = a \; .$$

Thus $a^{1/n}$ is to be defined as some solution of the equation

$$x^n = a \; .$$

When n is odd, the function $x \longrightarrow x^n$ is continuous and monotone and ranges over all real numbers. It follows that $x^n = a$ has precisely one solution, the n-th root of a which we have denoted by $\sqrt[n]{a}$ (Section A2-1). When n is even, $n = 2m$, the function $x \longrightarrow x^{2m}$ is only piecewise monotone, decreasing for negative values and increasing for positive values of x . The range of $x \longrightarrow x^{2m}$ is the set of all nonnegative numbers and every positive number appears as a value of x^{2m} twice, once for a positive value of x and once for a negative value. In order to give the symbol $\sqrt[2m]{a}$ a definite unambiguous meaning for $a \geq 0$ we defined the 2m-th root of a as the nonnegative solution of $x^{2m} = a$. Finally, when a is negative $\sqrt[2m]{a}$ has no meaning as a real number; we avoid complications caused by this fact by requiring that a be positive in all subsequent discussions of a^z where z is real. With this restriction we define $a^{1/n}$ for $a > 0$ as the unique positive solution of $x^n = a$.

We have defined $a^{1/n}$ for $a > 0$ and positive integral n . We now define fractional powers in general by

$$a^{p/q} = (a^{1/q})^p = (\sqrt[q]{a})^p \qquad (a > 0)$$

where p and q are integers and q is restricted to be positive. The algebraic properties (1) - (3) still hold for this larger class of powers. (The proof is straightforward and is left to you as an exercise.) Consequently the concept of power function $x \longrightarrow x^n$, (x " 0) , first defined only for natural numbers n , has been generalized first to all integral powers, then to all rational powers. The derivation of the properties of these functions, their algebraic properties (1) - (3) , their monotone character, their differentiability, raises no serious difficulties (Exercises 8-1, Nos. 1 and 2).

Complicated questions first arise when we attempt to generalize the idea of power further to irrational exponents. It seems natural to define a^x for an irrational number x in terms of approximations to x by rational numbers. For example, we would expect to get successively better approximations to $3^{\sqrt{2}}$ by using successively better decimal approximations to the exponent, e.g., 3^1 , $3^{1.4}$, $3^{1.41}$, $3^{1.414}$. However, if we were to pursue this idea and define real powers as limits of rational powers, the proof that all the general properties of rational powers carry over to all real powers would require prolonged formal argument. Instead, let us assume for the present that these properties actually hold. We shall come back to the problem of defining a^x when we obtain a more convenient insight into its solution.

Once we have defined the powers a^r for each real r and each positive a we are free to introduce the power function $x \longrightarrow x^r$, $(x > 0)$ for any real exponent r . Furthermore, we may also consider the exponential function given by

$$E_a(x) = a^x ,$$

where the base a is positive and the domain is the set of all real numbers.

If E_a has a derivative, then

$$E_a'(x) = \lim_{h \sim 0} \frac{a^{x+h} - a^x}{h} .$$

Assuming the property $a^r a^s = a^{r+s}$ to hold for all exponents, we have

$$E_a'(x) = \lim_{h \sim 0} \frac{a^x a^h - a^x}{h} .$$

Factoring and recalling the limit theorem for a product we get

$$E_a'(x) = a^x[\lim_{h \sim 0} \frac{a^h - 1}{h}] = E_a(x)[\lim_{h \sim 0} \frac{a^h - 1}{h}] .$$

But $\lim_{h \sim 0} \frac{a^h - 1}{h}$ (we assume the limit exists) does not depend on x ; it is constant. In fact,

$$\lim_{h \sim 0} \frac{a^h - 1}{h} = E_a'(0) , \tag{4}$$

and consequently,

$$E_a'(x) = E_a(x)E_a'(0) . \tag{5}$$

(At this time we are not concerned about the value of the constant $E_a'(0)$. In this discussion we have assumed only that E_a is differentiable at the one point $x = 0$ and satisfies the "functional equation" $E_a(x + y) = E_a(x) + E_a(y)$. Compare Exercises 5-1, No. 4.)

We see then that the derivative or rate of change of an exponential function is proportional to the function itself. This fact is typical of unregulated growth and decay processes, and makes exponential functions the key to an understanding of many natural phenomena.

The inverse of the power function $x \longrightarrow x^r, (r \neq 0)$, is the power function $x \longrightarrow x^{1/r}$ with reciprocal exponent, so that by taking inverses of power functions we obtain nothing new. The inverse of the exponential function $x \longrightarrow a^x$ with base a , for $a \neq 1$ is a new kind of function, the logarithmic function with base a :

$$x \longrightarrow \log_a x \text{ , for } a > 0 \text{ and } x > 0 , \qquad (a \neq 1) ,$$

where $\log_a x$ is defined in terms of exponentials by

$$E_a(\log_a x) = x$$

or

$$a^{\log_a x} = x .$$

The properties of logarithmic functions follow from those of the exponentials, in particular, corresponding to the formula $a^x a^y = a^{x+y}$, we have on setting $u = a^x$, $v = a^y$,

$$\log_a(uv) = \log_a u + \log_a v . \tag{6}$$

To determine the derivative of the logarithmic function we employ Theorem 4-3 on the differentiation of inverse functions. The derivative $D_u \log_a u$ where $u = a^x$ is just the reciprocal of the derivative $D_x E_a(x)$ on the assumption that the latter derivative exists. From

$$D_u \log_a u = \frac{1}{E_a'(x)} \qquad \text{where } x = \log_a u ,$$

and from (5), we have

$$D_u \log_a u = \frac{1}{E_a'(0)E_a(x)} .$$

Since the exponential and logarithmic functions are inverses we conclude that

$$D_u \log_a u = \frac{c}{u}$$

where the constant is recognized from (4) as $c = \frac{1}{E_a'(0)}$. Thus, the derivative of $\log_a x$ is proportional to x^{-1} . We expect, then, for positive α and x that

$$\int_\alpha^x \frac{1}{t}\, dt = \log_a x - \log_a \alpha = \log_a(\frac{x}{\alpha})$$

for some as yet unspecified base a . For simplicity, then, we fix $\alpha = 1$ and define the function L by

$$L(x) = \int_1^x \frac{1}{t}\, dt$$

where we anticipate that $L(x) = \log_a x$ with the value of a to be fixed accordingly.

In this section we have used a heuristic development to treat arbitrary powers, exponentials, and logarithms; that is, we have used an argument aimed at discovering the truth about these matters without concern for the detailed confirmation needed for absolute conviction. This is often the way a mathematician develops a new area: he explores tentatively, not necessarily proving every point as he goes, but framing conjectures for which he has reasonable grounds for belief. In this way he often comes upon some unifying principle or simple fact which can then be used to justify logically and completely what had before been accepted only provisionally. Just so, in our discussion we have discovered that the derivative of a logarithm is proportional to $\frac{1}{x}$. In other words, a logarithm is an integral of a very simple function. This fact now can be taken as a springboard for the development of the entire subject. We shall define the logarithm as an integral of $\frac{1}{x}$ and, with information of Chapters 6 and 7 about integrals, we shall be able to derive all the properties of logarithms, exponentials and powers encountered in our heuristic discussion.

Exercises 8-1

1. Prove that properties (1) - (3) hold for rational exponents, $a > 0$.
2. Establish the monotone character and differentiability of the power functions $x \longrightarrow x^r$ $(x > 0$, r rational).
3. Let f be defined for all real numbers and let f satisfy the functional equation

   (1) $$f(x + y) = f(x)f(y) \text{ , for all } x \text{ and } y .$$

   (a) Prove that if f is a solution of Equation (1) then either

   $$f(0) = 0 \quad \text{or} \quad f(0) = 1 .$$

   (b) Prove that if $f(0) \neq 0$, then there is no value of x for which $f(x) = 0$.
4. Let f satisfy the functional equation

   (1) $$f(xy) = f(x) + f(y)$$

   for all x , y in its domain.

   (a) Prove that the function $f : x \longrightarrow 0$ is the only solution of (1) that is defined for all real numbers x .

   (b) Prove that if f is a solution of (1) and the domain of f includes 1 and -1 but not 0 , then $f(1) = 0$ and f is an even function.

(c) Prove that if f is a solution of (1) then $f(x^r) = rf(x)$, r rational.

(d) Prove that if f is a solution of (1) and if f is differentiable at each $x \neq 0$, then $f'(x) = \frac{f'(1)}{x}$ for each $x \neq 0$.

(e) Using (d) show that any solution of (1) which is differentiable at each $x \neq 0$ is integrable on any closed interval $[a,b]$, where $ab > 0$.

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8-2. The Logarithm as an Integral.

Pursuing the lead of the preceding section we introduce the function L given by

$$L(x) = \int_1^x \frac{1}{t}\,dt\ , \qquad (x > 0), \tag{1}$$

and embark on the project of showing that L is actually a logarithmic function in the sense of Section 8-1.

Since $\frac{1}{t}$ is continuous for $t > 0$ the integral of (1) exists, and by the Fundamental Theorem of Calculus

$$L'(x) = \frac{1}{x}\ , \qquad (x > 0)\ . \tag{2}$$

Since the derivative L' exists it follows that L is continuous and, since $L'(x) > 0$ on the domain of positive x, that L is an increasing function. Further, since

$$L(1) = \int_1^1 \frac{1}{t}\,dt = 0 \tag{3}$$

it follows that $L(x)$ is negative for $0 < x < 1$ and positive for $x > 1$.

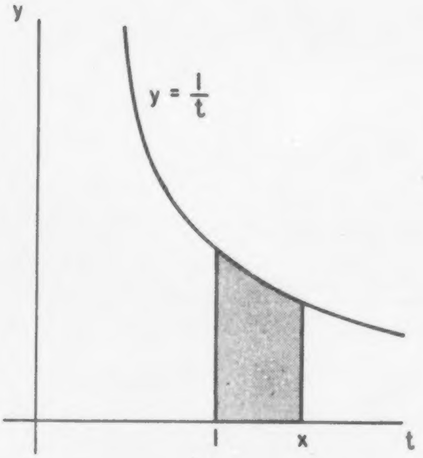


Figure 8-2a

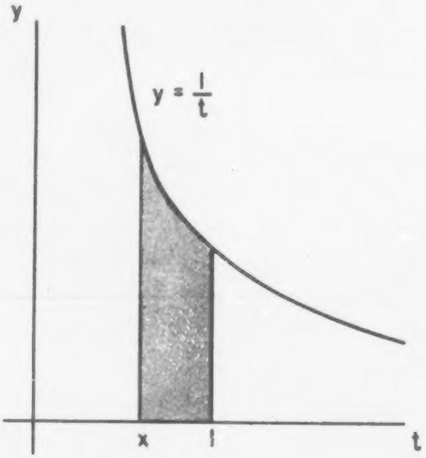


Figure 8-2b

Thus, in Figure 8-2a for $x > 1$, $L(x)$ is the area of the shaded region under the graph $y = \frac{1}{t}$ and in Figure 8-2b for $x < 1$ it is the negative of the area of the shaded region under the graph.

We observe for the sign of the second derivative that

$$L''(x) = -\frac{1}{x^2} < 0 , \qquad (x > 0) .$$

It follows that the graph $y = L(x)$ is flexed downward.

Next we prove that L satisfies the same addition property as logarithms.

THEOREM 8-2. The function L satisfies the equation

$$(4) \qquad L(ab) = L(a) + L(b)$$

for all values a and b in its domain.

Proof. If we consider $L(ab)$ and $L(b)$ we see that they differ by the constant $L(a)$ and should have the same derivative with respect to b
In fact, by the Chain Rule and Equation (2),

$$D_tL(at) = aL'(at) = a(\frac{1}{at}) = \frac{1}{t} ,$$

whence,

$$D_tL(at) = D_tL(t) .$$

If we integrate with respect to t from $t = 1$ to $t = b$ we obtain by the Fundamental Theorem

$$L(ab) - L(a) = L(b) - L(1) .$$

The addition property (4) follows at once from (3) .

As an immediate consequence of Theorem 8-2 we obtain a rule having the same form as that for the logarithm of a quotient:

Corollary 1. $\qquad L(\frac{a}{b}) = L(a) - L(b) .$

Proof. From Theorem 8-2, we have

$$\begin{aligned} L(a) &= L(\frac{a}{b} \cdot b) \\ &= L(\frac{a}{b}) + L(b) , \end{aligned}$$

from which the result follows immediately.

The proofs of the following corollaries are left as exercises.

Corollary 2. For all integers n,

$$L(a^n) = nL(a) .$$

Corollary 3. For all rational values r

$$L(a^r) = rL(a) .$$

With these results we are equipped to plot the graph of L and to examine some of its properties. Using estimation by upper and lower sums (or the more refined methods discussed in the chapter on numerical methods) we can obtain

$$L(2) = 0.693\ldots .$$

To calculate the values of L at other points we simply use the results above, e.g.,

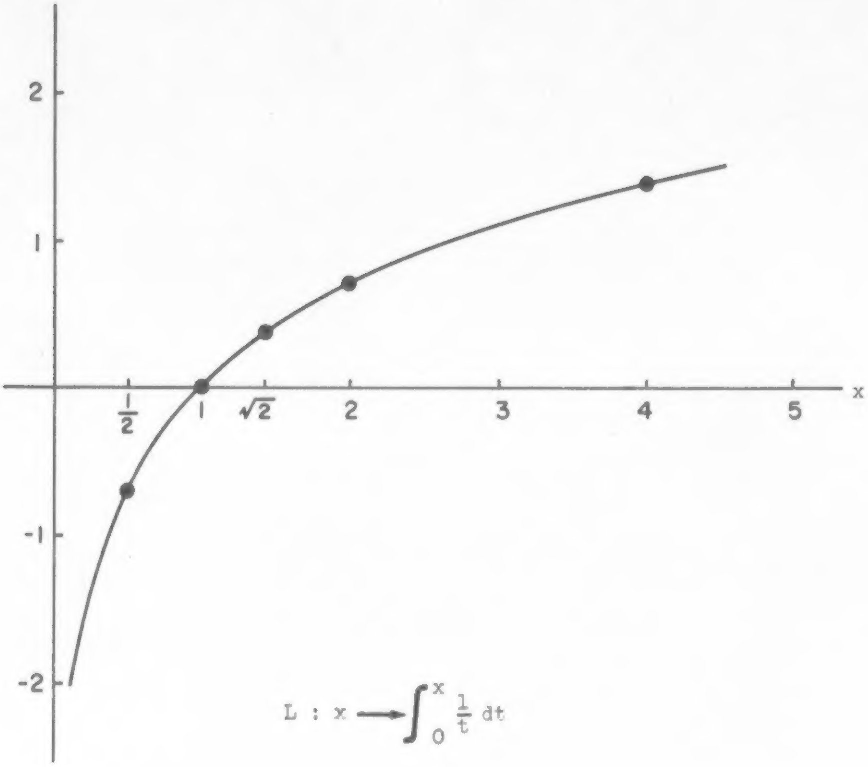
$$L(\tfrac{1}{2}) = -L(2) = -0.693\ldots ,$$

$$L(4) = 2L(2) = 1.386\ldots ,$$

$$L(\sqrt{2}) = \tfrac{1}{2}L(2) = 0.346\ldots , \text{ etc.}$$

By such means we can plot an adequate graph of L (Figure 8-2c).

Since $L(2^n) = nL(2)$ we see that the values of $L(x)$ have no bound either from above or below: by taking n positive and sufficiently large we can make $L(2^n)$ larger than any given positive number; by taking n negative and absolutely large we can make $L(2^n)$ less than any given negative number. Since L is continuous, it follows from the Intermediate Value Theorem that $L(x)$ ranges over all real values.



$L : x \longrightarrow \int_0^x \frac{1}{t}\,dt$

Figure 8-2c

Exercises 8-2

1. Prove Corollaries 2 and 3 to Theorem 8-2.

2. (a) Show that the area A of the standard region of $f : x \longrightarrow \frac{1}{x}$ over the interval $[1,2]$ satisfies the following inequality:

$$\frac{1}{3} + \frac{1}{4} < A < \frac{1}{2} + \frac{1}{3}$$

   (b) Approximate A to the nearest $\frac{1}{10}$. (You need not carry out the calculation; i.e., represent A as a sum.)

3. (a) For $a > 1$ show that the area A of the standard region of $f : x \longrightarrow \frac{1}{x}$ over $[1,a]$ satisfies the inequalities

$$1 - \frac{1}{a} < A < a - 1 .$$

(b) For $0 < a < 1$ show that A satisfies

$$1 - a < A < \frac{1}{a} - 1 .$$

4. (a) Make a careful plot of the graph of L using $L(2) = 0.693$ and interpolating further values between those already given.

(b) Draw the tangent to the graph of L at each of the points $x = \frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8$. Do these tangents conform to the plot in part a ?

(c) Obtain the solution of the equation $L(x) = 1$ with the accuracy available from your plot.

5. (a) Using estimates by means of upper and lower sums, how many values of the integrand $\frac{1}{x}$ are needed to calculate $L(2)$ within the tolerance indicated by $L(2) = 0.693$?

(b) Using the method of approximating $\frac{1}{t}$ by a linear function (see Sections 5-3(ii) and 5-7) on each interval of a subdivision of $[1,2]$ show how to estimate $L(2)$ and determine the number of values of $\frac{1}{x}$ needed to give $L(2)$ accurately to three decimal places.

6. (a) Starting from

$$\frac{1}{1 - x} = 1 + x + x^2 + \dots + x^{m-1} + \frac{x^m}{1 - x} ,$$

show that

$$-L(1 - x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \dots + \frac{x^m}{m} + R_m ,$$

where

$$R_m = \int_0^x \frac{y^m}{1 - y} . dy .$$

(b) For the interval $0 \le x < 1$, show that

$$0 \le R_m \le \frac{x^{m+1}}{(1 - x)(m + 1)} ;$$

and for $-1 \le x \le 0$ show that

$$|R_m| \le \frac{|x|^{m+1}}{m + 1} .$$

(c) Show that

$$\frac{1}{2}L\left(\frac{1+x}{1-x}\right) = x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots + \frac{x^{2n-1}}{2n-1} + R_n^* ,$$

$$|x| < 1 \text{ , where } |R_n^*| \leq \frac{x^{2n}}{2n} \cdot \max\left\{\frac{1}{1-x}, 1\right\} .$$

(d) Using part (c), obtain a value for $L(2)$ accurate to four decimal places.

(e) By setting $x = \frac{1}{2}$ in Part (c), we could obtain a value for $L(3)$. However, it is more efficient to calculate $L(3)$ by setting $\frac{1+x}{1-x} = \frac{3}{2}$ and using both Parts (c) and (d). Calculate $L(3)$ this way accurate to four decimal places.

(f) Obtain a value for $L(5)$ by setting $\frac{1+x}{1-x} = \frac{5}{4}$.

(g) Give several ways of obtaining a value for $\log 11$. Which is most efficient?

7. (a) Using upper and lower sums as estimates for $\int_1^n \frac{1}{t}\,dt$, obtain the inequality

$$\frac{1}{n} + L(n) < 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} < 1 + L(n) .$$

(b) Estimate $\sum_{n=1}^{10^{100}} \frac{1}{n}$.

(c) Prove that $\lim_{N\to\infty} \sum_{n=1}^{n} \frac{1}{n}$ does not exist.

8. Find $D_x L(|ax + b|)$ and obtain the corresponding integration formula over any interval $[h,k]$, where $ax + b \neq 0$ for any x in $[h,k]$.

8-3. The Exponential Function. General Powers.

We have verified that $L(x)$ defined as an integral of $\frac{1}{x}$ has properties of a logarithm, but the story is not yet complete. We must show that L is the inverse of an exponential function $x \longrightarrow a^x$ for some value a $(a \neq 1)$, and this in turn must be compatible with a definition of a^x for irrational exponents as a continuous extension of the function defined for rational exponents.

Since L is an increasing function it has an inverse

$$E : L(z) \longrightarrow z \qquad (z > 0) . \tag{1}$$

Since the domain of E is the range of L, the domain of E is the set of all real numbers. Since the range of E is the domain of L, the range of E is the set of all positive numbers. We obtain the graph $y = E(x)$, as for inverse functions in general, by reflecting the graph $y = L(x)$ in the line $y = x$ (Figure 8-3).

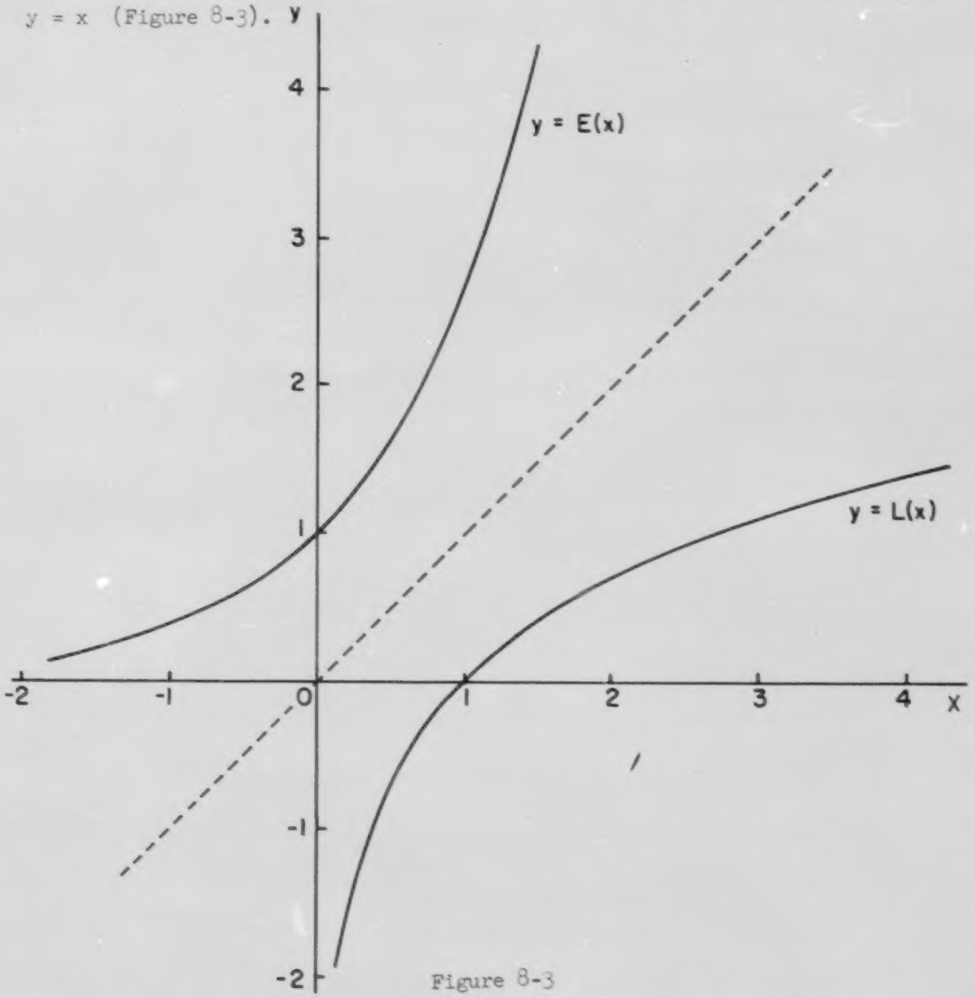


Figure 8-3

The properties of L can immediately be translated into properties of E. Since L is continuous and increasing, so also is E. Putting $L(1) = 0$ in (1) we obtain

$$E(0) = 1 . \tag{2}$$

Setting $u = L(a)$, $v = L(b)$ in Theorem 8-2 we obtain

$$u + v = L(ab) ,$$

and consequently,

$$E(u + v) = ab = E(u)E(v) . \tag{3}$$

It follows that

$$E(u - v) = \frac{a}{b} = \frac{E(u)}{E(v)} . \tag{4}$$

Furthermore, for any rational exponent r,

$$[E(u)]^r = E(ru) . \tag{5}$$

The details of verification for Equations (3) - (5) are left as an exercise.

For $a > 0$ and r rational, we obtain from (5)

$$a^r = \Big(E(L(a))\Big)^r = E(rL(a)) ,$$

where we recall that $E(u) = a$ and $u = L(a)$. This formula is particularly interesting because its left side has been defined only for rational values of r, whereas its right side is defined for all real values of r. It is therefore natural to use this formula to extend the definition of a^r to irrational values. In this way we fill the gap of Section 8-1 in the extension of the power functions to powers with real exponents.

DEFINITION 8-3. The power a^x is defined for all real values x and all positive values a by

$$a^x = E(xL(a)) .$$

Since E is continuous it is clear that

$$\begin{aligned} \lim_{r \sim x} a^r &= \lim_{r \sim x} E(rL(a)) \\ &= E\Big(\lim_{r \sim x} rL(a)\Big) && \text{(from Theorem 3-6e)} \\ &= E(xL(a)) && \text{(from Theorem 3-4b)} \\ &= a^x . \end{aligned}$$

In this way, we have established the continuity of the exponential function $x \longrightarrow a^x$ as an extra dividend. Furthermore, if r is restricted to rational values we see that the definition of a^x fulfills the condition that powers with irrational exponents shall be the appropriate limits of powers with rational exponents. Nonetheless, we shall be completely satisfied only if we can verify the laws of exponents, Section 8-1, Equations (1) - (3), for the more general class of powers. For the proof of the first law, we have

$$\begin{aligned} a^x a^y &= E(xL(a))E(yL(a)) && \\ &= E(xL(a) + yL(a)) && \text{from Equation (3)} \\ &= E((x + y)L(a)) && \\ &= [E(L(a))]^{x+y} && \text{from Equation (5)} \\ &= a^{x+y} && \text{from Equation (1).} \end{aligned}$$

The proofs of the two remaining laws are left as exercises.

The monotone property of the power function x^α is easily established. Since E and L are increasing functions we have for $0 < x < y$, using

$$y^\alpha - x^\alpha = E(\alpha L(y)) - E(\alpha L(x)) \ ,$$

that

$$y^\alpha - x^\alpha \begin{cases} > 0 \ , & \text{if } \alpha > 0 \\ = 0 \ , & \text{if } \alpha = 0 \\ < 0 \ , & \text{if } \alpha < 0 \ . \end{cases}$$

In words, x^α is increasing for positive α, constant (and equal to 1) for $\alpha = 0$, decreasing for negative α.

Finally, we verify the continuity of the power function x^α for any exponent α, rational or irrational. We have

$$\begin{aligned} \lim_{x \to \xi} x^\alpha &= \lim_{x \to \xi} E(\alpha L(x)) && \\ &= E\left(\lim_{x \to \xi} \alpha L(x)\right) && \text{(from Theorem 3-6e)} \\ &= E(\alpha L(\xi)) && \text{(from Theorem 3-4b)} \\ &= \xi^\alpha \ . && \end{aligned}$$

A simple proof suffices to establish the strongly monotone character of the exponential function $x \longrightarrow a^x$ for positive a when $a \neq 1$; it is left as an exercise. Once it is established that the function

$$x \longrightarrow a^x \qquad (a \neq 1)$$

is a strongly monotone function, we may introduce the inverse function

$$\log_a : a^z \longrightarrow z \qquad (a \neq 1) .$$

From Definition 8-3 we have

$$L(a^z) = zL(a)$$

from which it follows that

$$\log_a(a^z) = \frac{L(a^z)}{L(a)} .$$

We conclude that

(6) $$\log_a x = \frac{L(x)}{L(a)} .$$

To complete the discussion, we must show that E is itself an exponential function with a definite base. We set

$$e = E(1) .$$

Thus e is the unique value for which

(7) $$L(e) = 1 .$$

We obtain from Definition 8-3

$$e^x = E(xL(e)) = E(x) .$$

The constant e defined by Equation (7) is one of the important numbers of analysis; it appears in an astonishing variety of contexts, many of them seemingly quite remote from the ideas being discussed here. The value of e is given approximately by

$$e = 2.718$$

(Exercises 8-2, No. 4(c)). The function $E : x \longrightarrow e^x$ is called the exponential function in distinction to all other exponential functions. The exponential function E is often denoted by exp. The inverse function L is now definitely established as a logarithm

$$L : x \longrightarrow \log_e x .$$

The function L is referred to in this text and in all more advanced works as the logarithmic function and denoted simply by log without subscript. Common logarithms (logarithms with base 10) are still useful for hand computation but with the advent of machine computation they have lost much of their once great importance.* The logarithms used in analysis are almost invariably logarithms with base e .

For many purposes it is essential to have some idea of the relative orders of magnitude of the power, logarithmic and exponential functions. We show that any power function with positive exponent increases more rapidly than the logarithmic function and more slowly than the exponential function.

Lemma 8-3. For each positive α ,

$$\lim_{x\to\infty} \frac{\log x}{x^\alpha} = 0$$

and

$$\lim_{x\to\infty} \frac{x^\alpha}{e^x} = 0 .$$

Proof. We begin by obtaining estimates for $\log x$. For $1 \leq t \leq x$, we have $\frac{1}{x} \leq \frac{1}{t} \leq 1$, whence,

$$0 \leq \frac{x-1}{x} \leq \log x \leq \int_1^x \frac{1}{t}\,dt \leq x - 1 \leq x .$$

Thus, $\log x \leq x$ for $x \geq 1$ and, consequently, $\log \sqrt{x} \leq \sqrt{x}$, whence

$$\log x = 2 \log \sqrt{x} \leq 2\sqrt{x} .$$

We conclude that

$$0 \leq \frac{\log x}{x} \leq \frac{2}{\sqrt{x}} .$$

---

*John Napier (1550-1617) is justly regarded as the inventor of the logarithmic function. Although the basic idea was definitely "in the air" of his times, he was the first to publish a table of a logarithmic function (1614) and his ideas about logarithms were more insightful and efficient for the construction of tables than those of his contemporaries. Napierian logarithms are logarithms to the base e .

Henry Briggs (1561 - 1631) was largely responsible for the introduction of logarithms with base 10 for the purposes of computation.

Gregorius a Sancto Vincentio, S.J. (1584-1667) made the remarkable discovery of the addition property (Theorem 8-2) for $\log x$ interpreted as the area of the standard region under the graph of a hyperbola based upon its asymptote--this before systematic development of the calculus.

It follows from the Squeeze Theorem for limits as x approaches infinity (compare Corollary 2 to Theorem 3-4f) that

(8) $$\lim_{x\sim\infty} \frac{\log x}{x} = 0 .$$

This is the result from which the other order of magnitude estimates follows.

For $\alpha > 0$ we obtain

$$\lim_{x\sim\infty} \frac{\log x}{x^{\alpha}} = \lim_{x\sim\infty} \frac{1}{\alpha}\,\frac{\log x^{\alpha}}{x^{\alpha}} = \lim_{y\sim\infty} \frac{1}{\alpha}\,\frac{\log y}{y} = 0 ,$$

where $y = x^{\alpha}$.

Now we compare the values of power and exponential functions:

$$\frac{x^{\alpha}}{e^{x}} = \left(\frac{x}{e^{x/\alpha}}\right)^{\alpha} = \left(\frac{\log z}{z^{1/\alpha}}\right)^{\alpha}$$

where $z = e^{x}$. Employing the preceding result we obtain

$$\lim_{x\sim\infty} \frac{x^{\alpha}}{e^{x}} = \lim_{z\sim\infty} \left(\frac{\log z}{z^{1/\alpha}}\right)^{\alpha} = 0 .$$

Exercises 8-3

1. Verify properties (3) - (5) .
2. Prove Equations (2) and (3) of Section 8-1 for real powers in general.
3. Prove that the exponential function $x \longrightarrow a^{x}$, for positive a , $a \neq 1$, is strongly monotone.
4. Find the largest possible domain for each of the following functions.
   (a) $f : x \longrightarrow EL(x)$
   (b) $f : x \longrightarrow LE(x)$
   (c) $f : x \longrightarrow EE(x)$
   (d) $f : x \longrightarrow LL(x)$
5. Sketch the graph of the function given by $f(x) =$

(a)	2^{x}	(f)	$2^{x} + 2^{-x}$
(b)	2^{-x}	(g)	$2^{x} - 2^{-x}$
(c)	2^{x-1}	(h)	$2^{x+1/x}$
(d)	$2^{1/x}$	(i)	$2^{1/x} + 2^{-1/x}$
(e)	$2^{-1/x}$	(j)	$2^{1/x} - 2^{-1/x}$

6. Sketch the graph of

   (a) $f : x \longrightarrow EE(x)$,
   (b) $f : x \longrightarrow LL(x)$.

7. In each of the following solve for x in terms of y .

   (a) $y = L(\tan 2x)$

   (b) $y = E(x^2 + 1)$

   (c) $y = E(x - L(y))$

   (d) $y = \log (x - \sqrt{x^2 - 1})$

   (e) $y = E(x) - E(-x)$

8. Show that if $y = e^{\frac{1}{1-\log z}}$ and $z = e^{\frac{1}{1-\log x}}$, then $x = e^{\frac{1}{1-\log y}}$.

9. Prove that if f satisfies the functional equation $f(x + y) = f(x)f(y)$ for all x and y , and if $f(x) = 1 + x\,g(x)$ where $\lim_{x \to 0} g(x) = 1$, then $f'(x)$ exists for every x and $f'(x) = f(x)$.
   (See also Exercises 5-1, No. 4.)

8-4. Differentiation of the Logarithm and Related Functions.

The differentiability of the exponential and power functions follows at once from the differentiability of the logarithm. We have already found (Equation (2), Section 8-2)

(1) $$D_x \log x = \frac{1}{x} \qquad (x > 0) .$$

We apply the rule for differentiating inverses (Section 4-3) to obtain the derivative of the exponential function. If $y = E(x)$ we have

$$E'(x) = \frac{1}{L'(y)} = y = E(x) ;$$

that is,

(2) $$D_x e^x = e^x .$$

The exponential function has the remarkable property of being its own derivative. It is the prototype of functions describing unregulated processes of growth and decay in which a quantity changes at a rate proportional to the quantity itself. This property accounts for much of the great importance of the exponential function in mathematics and its applications.

We now obtain the derivative of the power function for any exponent, rational or irrational. From Definition 8-3, we have

$$x^a = E\Big(aL(x)\Big) , \qquad (x > 0) .$$

Applying the chain rule. we obtain

$$D_x x^a = E\Big(aL(x)\Big) \cdot \left(\frac{a}{x}\right) = x^a\left(\frac{a}{x}\right)$$

whence

(3) $$D_x x^a = ax^{a-1} .$$

In this way, we have obtained in all generality a result which we could prove earlier only for rational exponents (Section 4-3).

An exponential with any base a can be written in the form (Definition 8-3)

$$a^x = E\Big(xL(a)\Big).$$

It follows from the chain rule and Equation (2) that

$$\begin{aligned} D_x a^x &= E'\big(x\,L(a)\big)L(a) \\ &= E\big(x\,L(a)\big)L(a) \end{aligned}$$

whence

(4) $$D_x a^x = a^x \log a\ .$$

We recall from Section 8-1 (Equation 4) that for the function $E_a : x \longrightarrow a^x$

$$E_a{}'(0) = \lim_{h \to 0} \frac{a^h - 1}{h}\ .$$

Taking the derivative at $x = 0$ in (4) we obtain

$$D_x a^x \Big|_{x=0} = \log a\ ,$$

an interesting representation of $\log a$ as a limit:[1]

(5) $$\log a = \lim_{h \to 0} \frac{a^h - 1}{h}\ .$$

Exponential functions with bases other than e are not much used since any given exponential function is easily given in terms of exp by $a^x = e^{cx}$ where $c = \log a$.

Since a logarithm with any base a is simply proportional to the logarithm with base e (Section 8-3, Equation (6)),

$$\log_a x = \frac{\log x}{\log a}$$

we have, at once,

$$D_x \log_a x = \frac{c}{x}\ , \qquad \text{where} \quad c = \frac{1}{\log a}\ .$$

It is for this reason that logarithms with base e are often called "natural" logarithms, natural in the sense that the choice $c = 1$ yields the simplest possible expression for the derivative.[2]

---

[1] Equation (5) could have served to define the logarithm, but the necessary analytical approach differs considerably from the one we have adopted here.

[2] In many intermediate texts, the symbol $\ln x$ is used for the natural logarithm, but its use in professional literature is rare.

The properties of logarithms may be used to simplify the differentiation of complicated products and powers. For example, consider the problem of differentiating a product

$$\phi(x) = f_1(x)f_2(x) \ldots f_n(x) .$$

We assume that the derivative is taken at a point where $f_k(x) > 0$, $k = 1$, ... , n . (If $f_k(x) < 0$ we can replace $f_k(x)$ by its negative and change the sign of ϕ accordingly.) We have

$$\log \phi(x) = \sum_{k=1}^{n} \log f_k(x)$$

and from

$$D_x \log \phi(x) = \frac{\phi'(x)}{\phi(x)}$$

obtain

$$\phi'(x) = \phi(x)\, D_x \log \phi(x)$$

and, thence

$$\phi'(x) = \phi(x) \sum_{k=1}^{n} \frac{f_k'(x)}{f_k(x)} .$$

Example 8-4. To differentiate

$$\phi(x) = \frac{(x^2 + 1)^{3/2}\,(1 + \sin^4 x)^{25}}{2^x(x^2 + x + 1)}$$

we first obtain

$$\log \phi(x) = \frac{3}{2}\log(x^2 + 1) + 25\log(1 + \sin^4 x) - x \log 2 - \log(x^2 + x + 1) ,$$

then differentiate, and find

$$\frac{\phi'(x)}{\phi(x)} = \frac{3x}{x^2 + 1} + \frac{100 \sin^3 x \cos x}{1 + \sin^4 x} - \log 2 - \frac{2x + 1}{x^2 + x + 1}$$

which yields $\phi'(x)$ on multiplication by $\phi(x)$.

Exercises 8-4

1. Use logarithmic differentiation to evaluate each of the following.

   (a) $D_x\left(\dfrac{(2x+3)^4}{(3x^2-1)^{1/2}}\right)$

   (b) $D_x\sqrt[4]{\dfrac{x^4-1}{1+x^3}}$

   (c) $D_x\left(\dfrac{\cos^3 x}{\sqrt[3]{1+2\sin x}}\right)^{1/2}$

   (d) $D_x\left(\sqrt{x^2-1}\ \sqrt[3]{1+6x^3}\right)$

2. Find $D_x y$.

   (a) $y = 3^x$

   (b) $y = (\frac{1}{2})^x$

   (c) $y = e^{ax}$

   (d) $y = \log(1+x^2)$

3. Find $D_x y$.

   (a) $y = (2^x)^2$

   (b) $y = 2^{x^2}$

   (c) $y = e^{\cos x}$

   (d) $y = e^{\log\sqrt{x}}$

   (e) $y = \log\sqrt{e^x}$

4. Differentiate.

   (a) $y = x\log x - x$

   (b) $y = e^x\sin x$

   (c) $y = \arctan e^x$

   (d) $y = \log(\cos^2 x)$

5. Differentiate.

(a) $y = x^x$

(b) $y = x^{1/x}$

(c) $y = (\frac{1}{x})^x$

(d) $y = (\sin x)^x$

(e) $y = x^{\cos x}$

(f) $y = e^{x^x}$

(g) $y = x^{x^x}$

(h) $y = x^{e^{-x}}$

(i) $y = \log \log(x)$

(j) $y = \log \log \log(x)$

6. Differentiate. (First simplify, if possible.)

(a) $y = \log (\sec x + \tan x)$

(b) $y = \frac{1}{2a} \log \frac{a - x}{a + x}$

(c) $y = \log (x + \sqrt{a^2 + x^2})$

(d) $y = \sec x \tan x + \log (\sec x + \tan x)$

(e) $y = \log \sqrt{x^3 + \sin^2(x^2 + e^{x^3})^3}$

7. Evaluate each of the following integrals.

(a) $\int_0^1 e^{2x} \, dx$

(b) $\int_1^3 \frac{2}{x} \, dx$

(c) $\int_{-1}^{1} e^{ax} \, dx$

(d) $\int_0^1 2^x \, dx$

(e) $\int_0^a 2x \, e^{x^2} \, dx$

(f) $\int_0^a \frac{2x}{1 + x^2} \, dx$

(g) $\int_e^{e^e} \frac{dx}{x \log x}$

8. Show that if $e^x + e^y = e^{x+y}$ then $y' = -e^{y-x}$.

9. Prove:

(a) $\lim\limits_{x \sim 0} \frac{\log(1 + x)}{x} = 1$.

(b) $\lim\limits_{x \sim 0} \frac{e^{cx} - 1}{x} = c$, c constant.

(c) $\lim\limits_{x \sim 0} \frac{2 \log (1 + x) - 2x + x^2}{x^3} = \frac{2}{3}$.

10. Assuming the existence of the limit in Equation (5), use the representation of $\log a$ as a limit to derive the addition property of logarithms (Theorem 8-2).

11. Approximate the difference between a^x and a^{x+h} $(a > 0$ and $a \neq 1)$ for large $|x|$ where $x < 0$.

12. Prove that $e^x > 1 + x$ for all $x \neq 0$ and use this result to derive the following inequalities.

    (a) $e^{-x} > 1 - x$, for all $x \neq 0$.

    (b) $e^x > \frac{1}{1 - x}$, for $x > 1$.

    (c) $e^x < \frac{1}{1 - x}$, for $x < 1$, $x \neq 0$.

    (d) $e^{-x} < \frac{1}{1 + x}$, for $x > -1$, $x \neq 0$.

13. Prove

$$\lim_{a \sim -\infty} e^a = 0$$

14. Examine the behavior of f for small $|x|$ and very large $|x|$. Use the information obtained to sketch the graph of f.

    (a) $f : x \longrightarrow \dfrac{1}{1 - e^{-1/x}}$

    (b) $f : x \longrightarrow \dfrac{e^{1/x}}{1 + e^{1/x}}$

15. The velocity v of a falling body at time t is given by

$$v = \frac{g}{k}(1 - e^{-kt})$$

where g and k are positive constants and the positive sense of motion is directed downward.

(a) Find the acceleration a at any time t. Show that $a = g - kv$.

(b) Find the distances that the body travels between $t = 0$ and $t = \tau$.

(c) If g and t remain fixed determine

$$\lim_{k \sim 0} v = \lim_{k \sim 0} \frac{g}{k}(1 - e^{-kt}).$$

(d) On the same set of axes, sketch the graphs of the functions $t \longrightarrow s$, $t \longrightarrow v$, and $t \longrightarrow a$.

8-5. The Differential Equations of e^x, sin x, cos x.

The exponential function is basic to analysis and its applications; its importance can hardly be overstressed. The property which gives the exponential function its unique place is invariance under the operation of differentiation: $D_x e^x = e^x$.

We remarked that the class of functions which are proportional to their derivatives is important in the study of growth and decay processes. For example, the rate of increase in weight of a bacterial colony under favorable conditions is proportional to the weight already present. Your cup of coffee cools and your cold lemonade (assuming no ice) gets warm at rates proportional to the difference in temperature between your drink and its surroundings. In Chapter 9 we shall discuss such phenomena in detail. We shall study the properties of such a function through its differential equation $y' = cy$. In the same spirit we can study periodic phenomena in terms of the differential equation for the sine and cosine functions and study the properties of the functions themselves through the differential equation.

THEOREM 8-5a. A function f given by $y = f(x)$ satisfies the differential equation

$$y' = cy \tag{1}$$

subject to the initial condition

$$f(0) = a \tag{2}$$

if and only if

$$y = f(x) = ae^{cx}. \tag{3}$$

Proof. It is immediate that the function defined by (3) satisfies the conditions (1) and (2). To complete the proof we must show that the solution (3) is unique, that no other function satisfies the differential equation and the initial condition.

Let $u = g(x)$ be any solution of (1) and (2). Since $e^z > 0$ for all z we may suppose u is given in the form $u = v\,e^{cx}$ where $v = g(x)e^{-cx}$. Entering this expression for u in the differential Equation (1) we obtain

$$\begin{aligned} D_x u = D_x[v\,e^{cx}] &= v'e^{cx} + cv\,e^{cx} \\ &= cu \qquad \text{(from (1))} \\ &= cv\,e^{cx}. \end{aligned}$$

It follows that $v'e^{cx} = 0$, hence that $v' = 0$ and that v is constant. Since $g(0) = a$ we must have $v = a$.

As a consequence of Theorem 8-5a we see that the exponential function $E : x \cdots e^x$ could have been defined originally as the solution of the differential Equation (1) which satisfies the initial condition $f(0) = 1$. Assuming the existence of such a solution, we might then define powers, logarithms and exponentials with any base in terms of e^x. The catch is the assumption of existence of the solution of the differential equation. A general proof of existence for solutions of differential equations requires separate treatment and would be a digression for us.* Nonetheless, the idea of defining a function as the solution of a differential equation has special interest for us because the circular functions $\sin x$ and $\cos x$, which have not been defined analytically, both satisfy the simple differential equation

$$y'' + y = 0 . \tag{4}$$

The differential Equation (4) is as important for analysis and its applications as the Equation (1) for the exponential functions. It is the prototype of the equations which describe periodic phenomena, the oscillations of a spring, the electromagnetic vibrations which we perceive as color, the pressure oscillations which we perceive as sound.

To see how $\sin x$ and $\cos x$ are characterized as solutions of (4) we need a uniqueness theorem like that of Theorem 8-5a.

THEOREM 8-5b. There exists at most one solution $y = f(x)$ of (4) which satisfies the initial conditions

$$f(0) = a , f'(0) = b . \tag{5}$$

---

*In this case existence happens to be a consequence of a special property of the differential Equation (1); it is separable (Chapter 10), but the solution of (1) as a separable equation leads back to the logarithm and the point of departure adopted in this text.

Proof. Let u and v be solutions of (4) which satisfy (5). We see that $y = u - v$ is a solution of (4) which satisfies the initial conditions

$$(6) \qquad f(0) = 0 \, , \; f'(0) = 0 \, .$$

On multiplying in (4) by y' we obtain

$$y''y' + y'y = \frac{1}{2} D[(y')^2 + y^2] = 0 \, .$$

It follows that

$$(7) \qquad (y')^2 + y^2 = C$$

where C is constant. From the initial condition (6), however, we conclude that C must be zero, hence, both terms on the left in (7) must be zero. We conclude that $y = u - v = 0$ and that $u = v$.

From this theorem we see that the sine can be defined as the unique solution $u = \phi(x)$ of (4) which satisfies the initial conditions

$$(8) \qquad \phi(0) = 0 \, , \; \phi'(0) = 1$$

and the cosine as the unique solution $v = \psi(x)$ of (4) which satisfies the initial conditions

$$(9) \qquad \psi(0) = 1 \, , \; \psi'(0) = 0 \, .$$

Observe that if u and v are solutions of (4), then so is the linear combination, $av + bu$; and the unique solution satisfying the general initial conditions (5) is the linear combination

$$(10) \qquad y = av + bu \, .$$

These ideas permit us to dispense with geometry in defining the sine and cosine, but before we can make this approach meaningful we must show that solutions u and v satisfying the initial conditions (8) and (9) do exist. A differential equation may not have a solution. Equation (7), for example, has no solutions when C is negative. To prove existence we work with the inverse functions.

Recalling the rule for differentiating the arc sine, we set

$$(11) \qquad x = g(u) = \int_0^u \frac{1}{\sqrt{1 - t^2}} \, dt \, .$$

This integral defines $g : u \longrightarrow \arcsin u$ as an increasing function of u

for $-1 < u < 1$ since the integrand is continuous and positive on the open interval $(-1,1)$. It follows that the inverse function

$$\phi : g(u) \longrightarrow u$$

is a continuous and increasing function. Since $g'(u) \neq 0$ anywhere on the interval $(1,1)$ we see that ϕ is differentiable, and by the rule for differentiating inverses

$$u' = \phi'(x) = \frac{1}{g'(u)} = \sqrt{1 - u^2} \tag{12}$$

where $x = g(u)$. Since the function on the right is differentiable, it follows that u' is differentiable and we obtain by the chain rule and Equation (12)

$$u'' = - \frac{uu'}{\sqrt{1 - u^2}} = -u \ .$$

(Since u is restricted to the open interval $(-1,1)$ the possibility of a zero divisor does not arise.) We see that u satisfies the differential Equation (4). The value $x = 0$ in (11) can only occur when $u = 0$ since g is an increasing function. It follows on setting $x = 0$ in (11) and (12) that $\phi(x)$ satisfies the initial condition (8).

It is true that (11) defines the function ϕ only on a neighborhood of $x = 0$, but the differential equation itself can be used to extend the solution to all values of x .

Since u is differentiable it follows from (4) that u'' is differentiable and on setting $v = u'$ that

$$v'' + v = D_x(u'' + u) = 0 \ .$$

Consequently, $v = \psi(x) = \phi'(x)$ is also a solution of (4). Furthermore, from (12) $\psi(0) = \sqrt{1 - [\phi(0)]^2} = 1$, and from (4) $\psi'(0) = \phi''(0) = \phi(0) = 0$. The function ψ is therefore the unique solution satisfying the initial conditions (9).

We may now abandon all our doubts; the solutions $\phi(x)$ and $\psi(x)$ exist and are unique. The familiar circular functions may now be defined by

$$\begin{cases} \sin : x \longrightarrow \phi(x) \\ \cos : x \longrightarrow \psi(x) \ . \end{cases} \tag{13}$$

We still have work to do. The familiar rules governing the circular functions must now be derived. For the most part this is a simple matter and is left to the exercises. The integral (11) for the arcsine yields definitions of $\sin x$ and $\cos x$ only on a neighborhood of the origin. It is necessary to show that the differential equation and initial conditions determine solutions defined on the domain of all real values. It is also necessary to prove that these solutions are periodic and to establish the role of the number π. The technical details of these proofs are given in Appendix 8.

Exercises 8-5

1. For each of the following find the function f which satisfies the given differential equation subject to the given initial conditions.

   (a) $y' = 2y$; $f(0) = 5$
   (b) $y'' + y = 0$; $f(0) = 2$, $f'(0) = -2$

2. Show that the function given by $y = f(x) = b \sin ax + d \cos ax$ satisfies the differential equation $y'' + a^2 y = 0$ and the initial conditions $f(0) = d$, $f'(0) = ab$.

3. Show that the function f given by

$$y = f(x) = e^x \sin x$$

   satisfies the differential equation

$$y'' - 2y' + 2y = 0$$

   and the initial conditions $f(0) = 0$, $f'(0) = 1$. Show also that the fourth derivative of $f(x)$ is proportional to $f(x)$.

4. A particle moves along a line for 2 hours so that its velocity at any time t is given by $v = \frac{1}{2} e^{3t}$, in miles per hour. Find the displacement at the end of 2 hours and the distance traveled during the last hour.

5. Derive Formula (11) for the inverse of ϕ directly from Equation (7).

6. Prove the identity

$$u^2 + v^2 = 1$$

   where $u = \phi(x)$ and $v = \psi(x)$ as defined in this section.

7. Prove that $u \leq \arcsin u \leq \dfrac{u}{1 - u^2}$ for $0 \leq u < 1$.

8. Prove $\lim\limits_{x \sim 0} \frac{\sin x}{x} = \lim\limits_{x \sim 0} \frac{\phi(x)}{x} = 1$.

9. Prove the rule for differentiating the cosine; that is,

$$\psi'(x) = -\phi(x) \ .$$

10. Prove that ϕ and ψ as defined in this section are differentiable to all orders.

11. Prove that the sine is an odd function and that the cosine is even; i.e., that $\phi(-x) = -\phi(x)$ and $\psi(-x) = \psi(x)$.

12. Prove the addition theorem for the sine:

$$\phi(a + b) = \phi(a)\psi(b) + \phi(b)\psi(a) \ .$$

(Hint: Use the fact that $\phi(a + x)$ is a solution of the differential Equation (4).)

13. State and prove the corresponding addition theorem for the cosine.

14. Interpret the constant C in Equation (7) in terms of the amplitude of oscillation.

15. Discuss the existence and uniqueness of the solution of the following initial value problem

$$D^2 f + \omega^2 f = 0 \ ; \ f(0) = a \ , \ f'(0) = b \ .$$

8-6. The Number e .*

The properties of e and the exponential function can be derived from simple estimates obtained in the same manner as the estimates of Section 7-5 for $\sin x$ and $\cos x$. We consider values of x in the closed interval $[0,\alpha]$. From the monotone property of the exponential function, we have

$$e^0 \leq e^x \leq e^\alpha .$$

Integrating we obtain

$$e^0 x \leq \int_0^x e^t dt \leq e^\alpha x$$

$$x \leq e^x - 1 \leq e^\alpha x ,$$

whence,

$$1 + x \leq e^x \leq 1 + e^\alpha x \qquad (0 \leq x \leq \alpha). \tag{1}$$

In general, if

$$f(t) \leq e^t \leq g(t) , \qquad (0 \leq x \leq \alpha),$$

we have

$$\int_0^x f(t)dt \leq \int_0^x e^t dt \leq \int_0^x g(t)dt , \qquad (0 \leq x \leq \alpha),$$

whence,

$$1 + \int_0^x f(t)dt \leq e^x \leq 1 + \int_0^x g(t)dt , \qquad (0 \leq x \leq \alpha). \tag{2}$$

Applying the result of (2) to (1) and integrating repeatedly, we obtain

$$1 + x + \frac{x^2}{2} + \frac{x^3}{2\cdot 3} + \dots + \frac{x^n}{n!} \leq e^x \leq 1 + x + \frac{x^2}{2} + \frac{x^3}{2\cdot 3} + \dots + \frac{e^\alpha x^n}{n!} , \qquad (0 \leq x \leq \alpha). \tag{3}$$

A similar result holds for negative values of x (Exercises 8-6, No. 1-b).

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*The number e is justly called the Euler number after Leonard Euler (1707-1783) who recognized its pervasive role in analysis and established many of its properties.

In three prodigious treatises Analysin Infinitorum (1748), Calculi Differentialis (1755), Calculi Integralis (1768-70), Euler opened and developed vast areas of analysis. The fertility of his imagination and the sheer magnitude of his work are unmatched. There is scarcely any area of mathematics without its "Euler's Theorem" or "Euler's Formula."

From Formula (3) we can easily obtain precise estimates for e. We observe first, on taking $x = \alpha = 1$ and $n = 3$ that

$$e \leq 1 + 1 + \frac{1}{2} + \frac{e}{6} .$$

From this inequality we obtain $\frac{5e}{6} \leq \frac{5}{2}$, whence $e \leq 3$. Entering this result in (3), we find

$$e^x = 1 + x + \frac{x^2}{2} + \ldots + \frac{x^n}{n!} + \epsilon(x) \qquad (x \geq 0) \tag{4}$$

where for the error term $\epsilon(x)$ we have

$$0 \leq \epsilon(x) \leq \frac{(3^x - 1)x^n}{n!} . \tag{5}$$

In particular, setting $x = 1$ we obtain

$$0 \leq \epsilon(1) \leq \frac{2}{n!} .$$

Since $n!$ increases extremely rapidly with n (thus $14! = 8.7 \times 10^{10}$, so that (4) gives e to 9-place accuracy when $n = 14$) and the k-th term of the sum for e is obtained from the preceding term upon division by k, it is easy to obtain the value of e to a large number of decimal places. We illustrate by calculating e accurately to nine places. We carry the calculation to eleven places to allow for the error in rounding off.

n	$\frac{1}{n!}$		
0	1.00000	00000	0
1	1.00000	00000	0
2	.50000	00000	0
3	.16666	66666	7
4	.04166	66666	7
5	.00833	33333	3
6	.00138	88888	9
7	.00019	84127	0
8	.00002	48015	9
9	.00000	27557	3
10	.00000	02755	7
11	.00000	00250	5
12	.00000	00020	9
13	.00000	00001	6
14	.00000	00000	1
Total	e = 2.71828	18284	6

Rounding off to nine places we obtain the easily memorized result

$$e = 2.71828\,1828 \ . \tag{6}$$

(A more accurate computation would show that the total above was actually accurate to all eleven places.) We leave as an exercise the problem of verifying that the sum of the error from rounding off and from cutting off the calculation at $n = 14$ is less than half a unit in the ninth place.

For a given x it is easily seen that we can bring the error $\epsilon(x)$ in the estimate for e^x below any given tolerance by taking n large enough. For this purpose we set $n = r + m$ and choose $r > 2x$. Setting $c = \frac{(3^x - 1)x^r}{r!}$, we obtain from (5)

$$\epsilon(x) \leq c\left(\frac{x}{r + 1}\right)\left(\frac{x}{r + 2}\right) \cdots \left(\frac{x}{r + m}\right) . \tag{7}$$

Since $\frac{x}{r + k} < \frac{x}{2x + k} < \frac{1}{2}$, $(k = 1 , 2 , \ldots , m)$, we have

$$0 \leq \epsilon(x) \leq \frac{c}{2^m} .$$

We conclude, then, that

$$e^x = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{x^k}{k!} \qquad (x \geq 0) . \tag{8}$$

From the estimate

$$0 \leq e - \left(1 + 1 + \frac{1}{2} + \frac{1}{3!} + \cdots + \frac{1}{n!}\right) \leq \frac{2}{n!}$$

it is easy to show that e is not a rational number. The number

$$\nu = (n - 1)!\left(1 + 1 + \frac{1}{2} + \frac{1}{3!} + \cdots + \frac{1}{(n - 1)!}\right)$$

is an integer. If $e = \frac{p}{q}$ where p and q are integers then the preceding estimate takes the form

$$0 \leq \frac{p}{q} - \left(\frac{\nu}{(n - 1)!} + \frac{1}{n!}\right) \leq \frac{2}{n!}$$

whence, for the integer $\mu = (n - 1)!\,p - \nu q$

$$\frac{q}{n} \leq \mu \leq \frac{3q}{n} .$$

If we let n be any natural number bigger than $3q$ we obtain

$$0 < \mu < 1 .$$

The assumption that e is rational leads to the false conclusion that there is an integer between 0 and 1 . It follows that e is not rational.

From $L'(x) = \frac{1}{x}$ we can obtain another representation of e as a limit. We have

$$\begin{aligned} L'(1) &= \lim_{h \to 0} \frac{\log(1 + h) - \log(1)}{h} \\ &= \lim_{h \to 0} \log(1 + h)^{1/h} , \end{aligned}$$

whence, since $L'(1) = 1$,

$$\log \lim_{h \to 0} (1 + h)^{1/h} = 1$$

where at the last line we have used the continuity of the logarithmic function. From the definition of e (Section 8-3, Formula (7)) we have the fundamental result

$$e = \lim_{h \to 0} (1 + h)^{1/h} . \tag{9}$$

If we restrict h to values $\frac{1}{n}$ where n is a natural number we obtain

$$e = \lim_{n \to \infty} (1 + \tfrac{1}{n})^n . \tag{10}$$

The quantity $(1 + \frac{1}{n})^n$ can be interpreted as the value at the end of one year of a deposit of one dollar left to acquire interest at an annual interest rate of 100% compounded n times a year. If the interest is compounded continuously, that is, if the interest is calculated as the limit in which the number n of interest periods approaches infinity, the value of the principal at the end of one year will be e dollars, \$2.72 . Disappointingly small, isn't it? (Exercises 8-6, No. 5a)

Since the graph of $x \longrightarrow \log x$ is increasing and flexed downward it follows that $h \longrightarrow (1 + h)^{1/h}$ is a decreasing function. (Exercises 8-6, No. 7). This result gives us another way of estimating e from above. For $h > 0$, $(1 + h)^{1/h}$ must be no greater than its limit, e , and for $h < 0$, $(1 + h)^{1/h}$ must be no smaller than its limit. In summary,

$$(1 + h)^{1/h} \le e \le (1 - h)^{-1/h} \qquad (0 < h < 1) . \tag{11}$$

Thus, setting $h = \frac{1}{2}$ we obtain

$$(\frac{3}{2})^2 \leq e \leq (\frac{1}{2})^2$$

or

$$2.25 \leq e \leq 4 \ .$$

The estimates (11) for e are not particularly useful for calculating e, but they have value in theoretical discussions.

Setting $h = \frac{1}{n}$ on the left and $h = \frac{1}{n+1}$ on the right in (11) we obtain

$$(1 + \frac{1}{n})^n \leq e \leq (1 + \frac{1}{n})^{n+1}$$

or

$$(12) \qquad (\frac{n+1}{n})^n \leq e \leq (\frac{n+1}{n})^{n+1} \ .$$

This estimate has extremely interesting consequences. Note for the product of the terms $(\frac{k+1}{k})^k$, $k = 1, \ldots, n$, that

$$(\frac{2}{1})(\frac{3}{2})^2(\frac{4}{3})^3 \ldots (\frac{n+1}{n})^n = \frac{(n+1)^n}{1 \cdot 2 \cdot 3 \cdot 4 \ldots n} = \frac{(n+1)^n}{n!} \ .$$

Since each factor $(\frac{k+1}{k})^k$, $k = 1, 2, \ldots, n$, in the product is no greater than e by (12), we have,

$$e^n \geq \frac{(n+1)^n}{n!}$$

whence

$$n! \geq \frac{(n+1)^n}{e^n} \geq (\frac{n}{e})^n (1 + \frac{1}{n})^n \ .$$

Since $(1 + \frac{1}{n})^n$ increases to the limit e, we see that it has its least value at $n = 1$ and therefore

$$n! \geq 2(\frac{n}{e})^n \ .$$

This formula gives us some idea of the prodigious rate of increase of $n!$; for large values of n it increases faster than any exponential a^n.

It is possible to obtain an upper estimate for $n!$ by using the right side of (12). We form the product of the terms $(\frac{k+1}{k})^{k+1}$ for $k = 1, 2, 3, \ldots, n$. From (12) it follows that

$$(\tfrac{2}{1})^2(\tfrac{3}{2})^3(\tfrac{4}{3})^4 \ldots (\tfrac{n+1}{n})^{n+1} = \frac{(n+1)^{n+1}}{n!} \geq e^n ,$$

whence

$$n! \leq \frac{(n+1)^{n+1}}{e^n} \leq \frac{n^{n+1}(1+\frac{1}{n})^{n+1}}{e^n} .$$

Since $(1+\frac{1}{n})^{n+1}$ decreases to its limit, its greatest value is attained at $n = 1$; hence $(1+\frac{1}{n})^{n+1} \leq 4$ and

$$n! \leq 4n(\tfrac{n}{e})^n .$$

In summary,

$$(13) \qquad 2(\tfrac{n}{e})^n \leq n! \leq 4n(\tfrac{n}{e})^n .$$

It is possible to improve the numerical factors in (13) by leaving out the early terms in each product, those for which the approximation to e is not particularly good. (See Exercises 8-6, No. 10). By means of subtler tech-techniques it is possible to do much better. There is a beautifully simple asymptotic representation $\phi(n)$ for $n!$, that is, a function ϕ such that

$$\lim_{n\to\infty} \frac{n!}{\phi(n)} = 1 ;$$

this is the famous Stirling's Formula,

$$\phi(n) = \sqrt{2\pi n}\,(\tfrac{n}{e})^n ,$$

which involves not only the constant e , but somewhat mysteriously, since present considerations seem totally unrelated to the circle, the constant π .

Exercises 8-6

1. (a) Using mathematical induction prove the result of Formula (3).
   (b) Obtain a result similar to Formula (3) for negative values of x and obtain an error estimate for your result like that of (5).
   (c) Use the result of b to prove (7) for negative values of x.

2. Fill in the details to show that Formulas (4) and (5) follow from Formula (3).

3. Prove that the value for e given by (6) is correct to the indicated number of decimal places; i.e., show that the error in (6) is less than one half unit in the last place.

4. Use (9) and (10) to evaluate each of the following limits.

   (a) $\lim\limits_{x \to \infty} \left(1 + \frac{2}{x}\right)^{x}$

   (b) $\lim\limits_{x \to \infty} \left(1 + \frac{1}{2x}\right)^{x/3}$

   (c) $\lim\limits_{h \to 0} (1 + 2h)^{3/h}$

   (d) $\lim\limits_{h \to 0} (1 + \sin h)^{2 \csc h}$

   (e) $\lim\limits_{h \to \infty} \left(1 - \frac{h}{2}\right)^{1/2h}$

5. Evaluate $\lim\limits_{x \to \infty} x(\sqrt[x]{n} - 1)$.

6. (a) A California savings and loan association offers an interest rate of 4.85% compounded continuously. What is the equivalent annual interest rate for money left on deposit one year?
   (b) How long does it take for an amount of money at the same interest rate (4.85% compounded continuously) to double itself?

7. (a) Prove that $x \rightarrow (1 + x)^{1/x}$ is a decreasing function.

(b) Which is greater

$$1000^{1001} \text{ or } 1001^{1000} \text{ ?}$$

(c) Which is greater

$$1{,}000{,}000^{1{,}000{,}001} \text{ or } 1{,}000{,}010^{1{,}000{,}000} \text{ ?}$$

∧8. Show that for $n \geq 9$, $\sqrt{n}^{\sqrt{n+1}} > \sqrt{n+1}^{\sqrt{n}}$.

9. Show how to obtain an approximation of

$$\int_0^t e^{x^2}\, dx \,, \qquad (0 < t < 1) \,.$$

Use Equation (4) to obtain an estimate of the error of approximation.

10. Prove that $n! > 3(\frac{n}{e})^n$ for $n > 1$.

[Hint: Use mathematical induction.]

8-7. The Hyperbolic Functions.

For reference we include here a brief discussion of the simple combinations of exponential functions known as hyperbolic functions. These functions have properties which parallel those of the circular functions. In analogy with the circular functions[*] we define the hyperbolic sine, cosine, and tangent, respectively, as follows:

$$\sinh x = \frac{e^{x} - e^{-x}}{2} ,$$

$$\cosh x = \frac{e^{x} + e^{-x}}{2} ,$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}} .$$

The analogies between the circular functions and the hyperbolic functions are exhibited in the following formulas (note carefully the differences in sign from the parallel formulas for the circular functions)

(1) $\cosh^2 x - \sinh^2 x = 1 .$

(2) $\sinh (x + y) = \sinh x \cosh y + \cosh x \sinh y .$

(3) $\cosh (x + y) = \cosh x \cosh y + \sinh x \sinh y .$

(4) $D \sinh x = \cosh x .$

(5) $D \cosh x = \sinh x .$

(6) $D \tanh x = 1 - \tanh^2 x .$

The derivation of these formulas is left as an exercise.

The principle features of these functions are easily described. The hyperbolic sine and tangent are odd functions, the hyperbolic cosine an even function. The intercepts of their graphs are given by $\sinh 0 = 0$, $\cosh 0 = 1$, $\tanh 0 = 0$. For all x , $\cosh x \geq 1$ and $|\tanh x| < 1$. Since

$$\lim_{x\sim\infty} e^{-x} = \lim_{x\sim -\infty} e^{x} = 0$$

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[*] In the theory of functions on the domain of complex numbers it is established that the circular functions are related to the exponential function by formulas similar to those of the hyperbolic functions.

cosh x approximates $\frac{1}{2}e^{x}$ from above and sinh x from below for large positive values of x . Similarly, cosh x approximates $\frac{1}{2}e^{-x}$, and sinh x , $-\frac{1}{2}e^{-x}$, for large negative values of x . The graph y = tanh x has the asymptotes y = 1 for large positive values of x , and y = -1 for large negative values of x . These features are depicted in Figure 8-7a. The demonstration of the properties of these functions is left as an exercise.

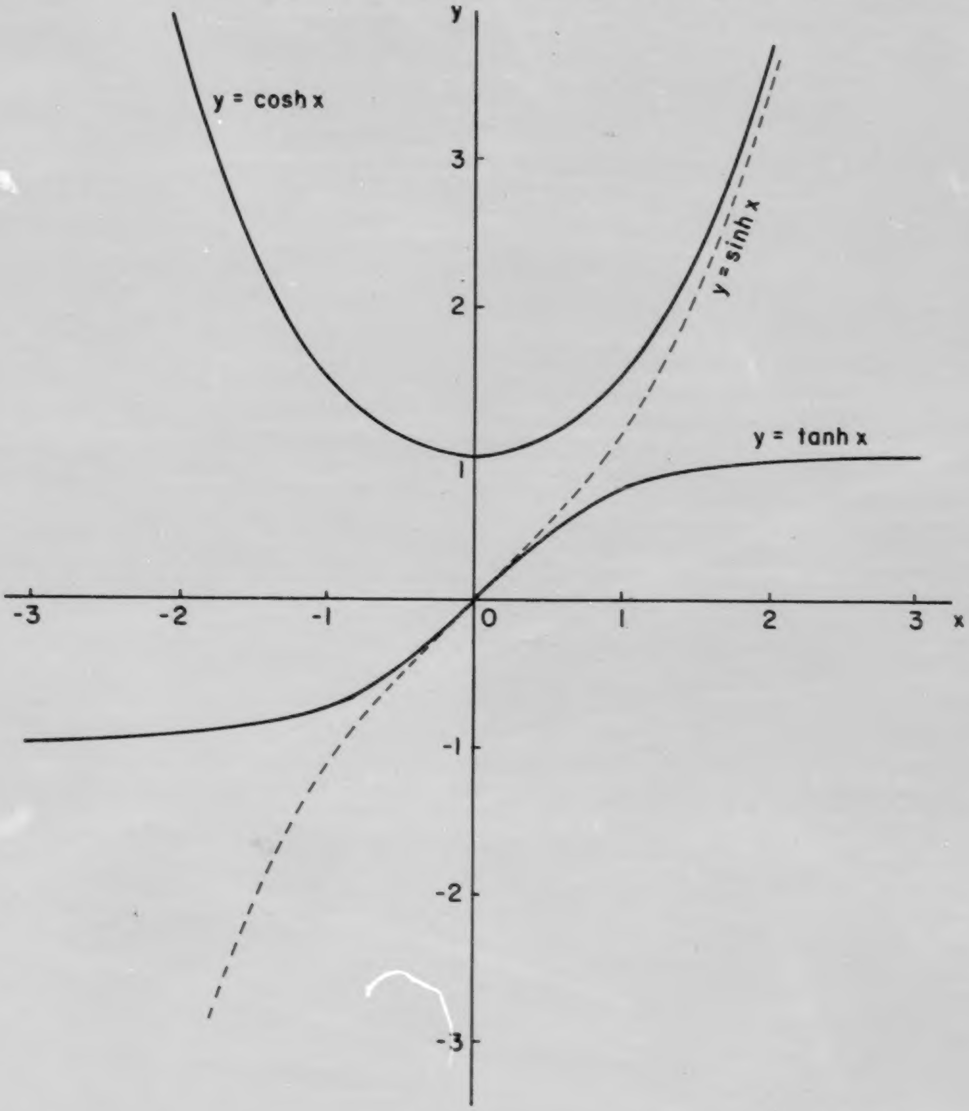

Figure 8-7a

Just as the circular functions are associated with the unit circle $x^2 + y^2 = 1$, the hyperbolic functions are associated with the rectangular hyperbola $x^2 - y^2 = 1$; the point (x,y) where $x = \cos\theta$ and $y = \sin\theta$ is a point on the circle; the point (x,y) where $x = \cosh u$ and $y = \sinh u$ is a point on the hyperbola (Figures 8-7b, c).

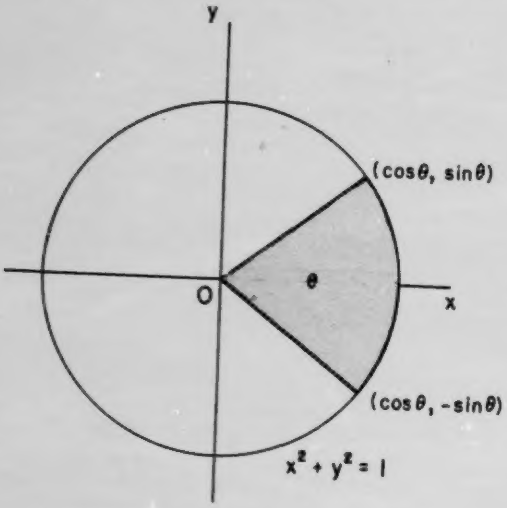

Figure 8-7b

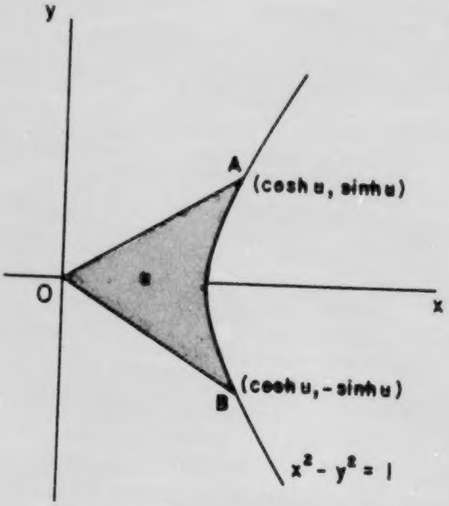

Figure 8-7c

There is a geometrical analogy between the parameter θ and the parameter u. In this analogy, we interpret θ not as an angle but as the area of a sector of the circle with central angle 2θ (shaded region in Figure 8-7b); as we shall see, the parameter u can be interpreted as the area of the corresponding region for the hyperbola (shaded region in Figure 8-7c).

Let α be the area of the hyperbolic sector in question. In terms of the area of the triangle AOB and the standard region under the graph $y = \sqrt{t^2 - 1}$ over $[1,x]$ we have

$$(7) \qquad \alpha = xy - 2\int_1^x \sqrt{t^2 - 1}\,dt$$

$$= \cosh u \sinh u - 2\int_1^{\cosh u} \sqrt{t^2 - 1}\,dt .$$

On differentiating with respect to u, we obtain

$$\begin{aligned} \frac{d\alpha}{du} &= \cosh^2 u + \sinh^2 u - 2\sqrt{\cosh^2 u - 1}\,\sinh u \\ &= \cosh^2 u + \sinh^2 u - 2\sinh^2 u \\ &= \cosh^2 u - \sinh^2 u \\ &= 1 . \end{aligned}$$

It follows that $\alpha = u + C$, and on taking $u = 0$ in (7) that the constant C is zero. We conclude that $\alpha = u$, or that u is the area of the hyperbolic sector.

Exercises 8-7

1. Derive Formulas (1) - (6) from the definitions of the hyperbolic functions.
2. Apply the methods of Section 5-8 to discuss the graphs of the hyperbolic functions.
3. Sketch the graphs of $y = \frac{1}{\cosh x}$ (that is, $y = \operatorname{sech} x$) and $y = \frac{1}{1 + x^2}$ on the same set of axis.
4. Find $\lim_{x \to 0} \frac{\sinh x}{x}$.
5. Differentiate.

   (a) $\frac{1}{\tanh^2 x}$

   (b) $\sinh (2 \log x)$

   (c) $e^x \sinh x$

   (d) $\frac{\cosh x + \sinh x}{\cosh x - \sinh x}$.
6. Show that $\sinh y + \cosh y = e^y$, for each y, and verify that $(\sinh x + \cosh x)^n = \sinh nx + \cosh nx$.
7. Show that $\arctan (\sinh x) = \arcsin (\tanh x)$.
8. Show that $D_x(2 \arctan e^x) = D_x \arctan (\sinh x)$.

   Does $2 \arctan e^x = \arctan (\sinh x)$?
   Justify your answer.
9. Calculate the inverse functions of sinh , cosh , and tanh in terms of logarithms.

10. Obtain the derivatives of the inverse hyperbolic functions using (1) and note how they differ from the derivatives of the corresponding inverse trigonometric functions. Observe that the derivatives of the inverse hyperbolic functions are algebraic functions.

11. Find the length of the catenary $y = a \cosh \frac{x}{a}$ between $x = 0$ and $x = b$.

12. Use (3) of Section 8-6 to find upper and lower polynomial bounds for $\sinh x$ and $\cosh x$.

13. (a) Obtain a formula for $\tanh (a + b)$ in terms of hyperbolic tangents.
    (b) Give $\tanh 4x$ in terms of $\tanh x$.

14. The differential equation

    (a) $D^2 f - f = 0$ is satisfied by cosh and sinh .

    Prove the uniqueness of the solution of (a) under the initial condition

    (b) $f(0) = a$, $f'(0) = b$

    and express the solution in terms of sinh and cosh .
    (Hint: Show that if f is a solution of (a) , then $g = Df - f$ is a solution of $Dg + g = 0$.)

Miscellaneous Exercises

1. Evaluate:

   (a) $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\frac{1}{n+k} = \lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\left(\frac{1}{1+\frac{k}{n}}\right)\frac{1}{n}$

   [Hint: Use Theorem 6-3c.]

   (b) $\lim\limits_{n\to\infty}\sum\limits_{k=0}^{n}\frac{1}{n+a+bk}$, $a, b \geq 0$.

2. Find the mean value (Exercises 6-4, No. 20) of the function

   $$f : x \longrightarrow \frac{1}{x}$$

   on the interval $[1,2]$.

3. If $F(x) = \int_0^{\sqrt{x}} e^{-t^2}\,dt$, $x > 0$, find $F'(x)$.

4. The region bounded by the curve $y = e^{-x}$ and the lines $y = 0$, $x = 0$, $x = 10$ is rotated about the x-axis. Compute to three significant digits the volume of the solid of revolution so generated. Justify any approximations you use.

5. Find all solutions of the equation $e^x = e$. Justify your answer.

6. Find an equation of the tangent line to the graph of $y = e^x$

   (a) at $(0,1)$.

   (b) at (a,e^a) .

   (c) that passes through the origin.

7. Find an equation of the only tangent line to the graph of $y = \log x$ that passes through the origin. Show that there is only one such line.

8. Find an equation of the tangent line to the curve $y = \log(\tan x)$ at the point of inflection (x_0,y_0) where $0 < x_0 < \frac{\pi}{2}$.

9. Find equations of the tangent line and normal line, respectively, to the graph of f at the given point.

   (a) $f : x \longrightarrow \dfrac{1}{1 + e^{-x}}$, at $x = 0$

   (b) $f : x \longrightarrow x^{x}$, at $x = \dfrac{1}{e}$

10. At any point where two curves intersect, the angle between their respective tangent lines is called their "angle of intersection." For each point of intersection of the graphs of $f : x \longrightarrow e^{-x^2}$ and $g : x \longrightarrow e^{-x}$ find the corresponding angle of intersection.

11. Show that the curve $y = \log|\sin 2x|$ is everywhere flexed downward. Sketch the curve.

12. Find the area of the standard region under the graph of f over the given interval.

    (a) $f : x \longrightarrow e^{x}$, $[-1,1]$

    (b) $f : x \longrightarrow e^{1-x}$, $[-1,1]$

    (c) $f : x \longrightarrow \tan x$, $[0, \frac{\pi}{3}]$

13. Find the volume of the solid of revolution generated by f on the given interval.

    (a) $f : x \longrightarrow e^{x}$, $[-1,1]$

    (b) $f : x \longrightarrow e^{1-x}$, $[-1,1]$

14. Derive a formula for the volume V of the solid of revolution generated by $f : x \longrightarrow a^{x}$ $(a > 0)$ on $[0,b]$.

15. Find the volume of the solid generated by revolving the region bounded by the curve $y = e^{-x/2}$, the x-axis, the y-axis and the line $x = 1$ about the line $y + 1 = 0$.

16. Sketch the graphs of each of the following, indicating extrema, points of inflection, intervals of downward and upward flexure.

    (a) $f : x \longrightarrow (\log x)^2 - x$

    (b) $f : x \longrightarrow \dfrac{x}{(\log x)^2}$

17. If A is a constant evaluate

$$\int_{\cosh^2 x}^{\sinh^2 x} A\,dt$$

18. Integrate x^2e^{mx} by assuming the integral has the form $(ax^2 + bx + c)e^{mx} + k$. Generalize this result.

19. (a) Verify that the derivative of

$$f(t) = (a \cos \quad t + b \sin \quad t)e^{kt}$$

is an expression of the same general form.

(b) Integrate $e^{ax} \sin bx$.

(c) Integrate $e^{ax} \cos bx$.

(d) Show that $f(t)$ in (a) is a solution of the differential equation

$$D^2f - 2kDf + (k^2 + \omega^2)f = 0 .$$

20. Establish the following limits:

(a) $\lim\limits_{x \sim 0} x^m \log(1 + x) = 0 , m > -1$.

(b) $\lim\limits_{x \sim \infty} x^m \log(1 + x) = 0 , m < 0$.

(c) $\lim\limits_{x \sim 0} x^p \log x = 0 , p > 0$.

21. Show

$$\lim_{x \sim a} \phi(x)^{\psi(x)} = L , L > 0 ,$$

if and only if

$$\lim_{x \sim a} \psi(x) \log \phi(x) = \log L .$$

22. Using the previous exercise, evaluate each of the limits in Exercises 8-6, Number 4.

∧23. Evaluate each of the following limits if it exists:

(a) $\lim\limits_{x \sim 0} \left(\frac{\sin x}{x}\right)^{1/x}$.

(b) $\lim\limits_{x \sim 0} \left(\frac{\sin x}{x}\right)^{1/x^2}$.

(c) $\lim\limits_{x \sim 0} \left(\frac{\sin x}{x}\right)^{1/x^3}$.

24. The formula for the normal probability curve used in statistics is

$$y = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-m)^2/2\sigma^2}$$

where m is the average (or mean) value of x and σ is the standard deviation and measures the spread of the curve. Find the extremum and points of inflection and sketch the curve. For simplicity let $m = 0$ and $\sigma = 1$ so that

$$y = f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} .$$

25. Determine the rectangle of largest area with base along the x-axis and two vertices on the graph of $f : x \longrightarrow e^{-x^2}$.

26. Prove that the function f given by

$$f(x) = \begin{cases} \frac{\sin x}{x} & , x \neq 0 \\ 1 & , x = 0 \end{cases}$$

(a) is continuous for all x ;

(b) is differentiable for all x .

Chapter 9

GROWTH, DECAY AND COMPETITION

9-1. Introduction.

"Mathematics enriches science by providing system and organization"--so we claim in the introduction. Yet until now we have exhibited the calculus only as an appropriate language in which a few of the concepts of the sciences may be couched. That it serves to reveal larger patterns of thought remains to be shown.

One way of demonstrating how mathematics organizes knowledge is to examine one of the mature areas of today's science, an area which has undergone a long historical development to reach its present systematic deductive stage. Such an area can be presented as a completely mathematical system with its own axioms and theorems. But this is only one aspect of mathematical thinking.

Mathematics plays a role in every stage of the development of a science into a deductive system--from the initial classification of related phenomena to the search for the least number of fundamental principles on which they depend--from the establishment of the laws of nature which serve as the axioms for a deductive system to the unfolding of the consequences implicit in such mathematical models of natural phenomena.. Science deals with phenomena--with observations, experiments, and measurements on nature. Special areas of science require special equipment, and special measurement techniques. All areas require mathematical thinking, mathematical tools and mathematical models.

All the sciences, from aerodynamics to zymology use mathematical models to organize the complicated phenomena observed in nature. To construct a model, we isolate the effects that appear to be fundamental, and we define relevant variables, parameters and functions. As suggested by our observations and measurements, we seek appropriate equations for the dependence of the functions on the essential variables. These will often take the form of differential equations completed with auxiliary conditions that specify, for example, the intial values of the functions and variables at the start of a process. The solutions of the equations subject to the auxiliary restrictions may then be compared with additional measurements to determine their domain of applicability in nature.

Underlying a bewildering variety of natural phenomena there seem to be but a few basic processes. The same mathematical models appear again and again in areas which bear no evident relation to each other. The equations are the same, but the functions and variables, like character actors who change their names but not their roles from play to play, represent different measurable quantities in each science. This is one way mathematics can organize knowledge, by revealing a common basic structure; a unity amid diversity.

In Chapter 15 we shall see how mathematics can be used to systematically develop a single area of science. But here we shall see how mathematics can cut across the sciences, how one mathematical model reveals a basic pattern which crops up in a multitude of a different contexts. We shall treat processes of "growth," "decay" and "competition" and see how the same basic process governed by the same differential equation is given different disguises in the different sciences. As we encounter such special instances of a general principle, we cannot help in retrospect but see a touch of the comic in how for each specialty we veil even the basic mathematical terms in esoteric labels, and a touch of the pathetic in how laboriously we redevelop the same basic mathematics within special desciplines through ignorance of the generality of the ideas.

The dissemination of a good story by word-of-mouth will be our primary example of a growth process. So we shall take up the threads of the Introduction (1-3) and see how the story of Helen of Troy may have been passed along until you heard of it--or how you could have heard about the calculus. We take up other threads of the Introduction and see how the basic mathematical model for the spread of stories is a pattern for other processes like that of remembering and forgetting isolated facts. We go on to show how the model may be altered to give more realistic descriptions of phenomena.

We are telling a story, a story about stories. We are not trying to teach science or mathematics, but to tell how the two are interrelated. Read our story through at one sitting. You should pay little attention to the specific fine points of the mathematics or the individual sciences.* The mathematics will be made precise in the exercises and the science may be learned at your own need and pleasure. We cannot always be amusing; the same equation which describes the broadcasting of a good joke serves also to describe the propagation of vicious gossip or the spread of an epidemic.

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*The exercises for this section are placed at the end to avoid interrupting the train of thought.

9-2. A Model for Growth. The Spread of a Story.

Once upon a time, (time t_0) I told a number (N_0) of friends a story about my good friend Al. Months later (time t) someone came up and asked "Did you hear the one about Al?" Since I had started the whole business, I didn't have to listen. Instead I asked myself, "How many $[N(t)]$ people have now heard the story about Al?"

How many people know the story about Al? Good stories spread, and this was a good one; the number of people who know it grows with time. The number $N(t)$ of people who know it at a time t should be proportional to the original number N_0 who were told the story at time t_0 --to the N_0 story tellers who couldn't keep a good thing to themselves. The older the story, the more people know it. Therefore $N(t)$ increases with the length of time $t - t_0$ that the story has been circulating, as well as with the number of people available to spread it. If $N(t)$ people know the story at time t, how many $N(\tau)$ know it at a slightly later time τ? It is plausible to expect that the number $N(\tau) - N(t)$ of people who learn the story in the interval $[t,\tau]$ is approximately proportional to both $N(t)$ and to the small time interval $\tau - t$. We accept these ideas as the initial assumptions, and express them mathematically in the form

$$N(\tau) - N(t) = A\, N(t)[\tau - t]\,, \qquad N(t_0) = N_0\,, \tag{1}$$

where A is a positive constant-- the growth coefficient. (Have we left out anything? Yes. We'll discuss that in a later section.)

Accepting (1) as an adequate model for the change in N over a small time interval still does not tell us how $N(t)$ is related to the initial value N_0. To determine this, we let the time interval approach zero and thus replace (1) by a differential equation, and then integrate over t to obtain $N(t)$ in terms of N_0.

From (1) we have

$$\frac{N(\tau) - N(t)}{\tau - t} = A\, N(t)\,. \tag{2}$$

If we discount the fact that friends come in integral packages (usually) and go to the limit as τ approaches t, we obtain

$$\frac{dN(t)}{dt} = A\, N(t)\,, \qquad N(t_0) = N_0\,. \tag{3}$$

Equation (3) states that the instantaneous rate of change of N is proportional to N: this is the basic equation for growth. Later on, we will also consider the case where A is negative; with A negative we have the basic equation for decay. (If A is zero, then N is a constant, and there is nothing to talk about--neither for potential story tellers nor for us.)

For convenience in all that follows we take $t_0 = 0$ as the original time. Thus (3) becomes

(4) $$\frac{dN(t)}{dt} = A\,N(t)\ , \qquad N(0) = N_0\ .$$

But the conditions of Theorem 8-5a are precisely those of (4). We conclude that

(5) $$N(t) = N_0 e^{At}\ , \qquad N_0 = N(0)\ ,$$

where t is the time that has elapsed since the start of the process.

From Equation (5) we see that N increases beyond any bound as t approaches infinity, which is not realistic for what we know about story telling (and other growth processes). Later on we consider a more realistic model. The present model is incomplete and should be restricted to moderately small time intervals.

We have told a story about stories to get to (5). Now that we have (5), we recognize that the result has other interpretations and that the analysis has other applications. Equation (5) provides an elementary model for the growth of timber and vegetation, the growth of populations (people, bacteria), the growth of money in banks (generous banks where they credit the interest to the capital instantaneously), the growth of a substance in the course of a chemical reaction, and so on.

We can now answer such questions as:

If I tell 2 people the story at $t = 0$, and if the constant of proportionality in (3) is $A = 1$, then how much is $N(t)$ at time $t = 7$ days? The answer from (5) is $2e^7$ or approximately 2193; thus more than 2000 people know the story a week after I started to spread it.

If I deposit $10 at 5% interest per year and the bank adds the interest to the original amount continuously, then when will it reach $20? For $A = \frac{5}{100}$, it follows from (5) that $\log(\frac{20}{10}) = (\frac{5}{100}) \cdot t$. Thus $t = 20 \log 2 = 20(0.693\ldots) \approx 13.9$ years.

9-3. Model for Decay.

(i) Radioactive disintegration. The same considerations that led us to our simple model for growth apply equally to the analogous model for decay. We take a negative constant proportionality $-A$ in (3) of Section 9-2 to correspond to $N(t)$ decreasing in time, and apply

$$(1) \qquad \frac{dN(t)}{dt} = -A\,N(t) \quad , \quad N(0) = N_0 \; ;$$

$$N(t) = N_0 e^{-At}$$

to the problem of radioactive decay. Different radioactive substances disintegrate at different rates corresponding to different values of the decay coefficient A . It is convenient to express the coefficient A in terms of the half-life of the substance, the time it takes half of the initial amount of substance to disappear. (Why not the whole-life?) If τ is the half-life then from (1) we have

$$\frac{N(\tau)}{N_0} = e^{-A\tau} = \frac{1}{2}$$

so that

$$(2) \qquad \tau = \frac{-\log \frac{1}{2}}{A} = \frac{\log 2}{A} \approx \frac{0.693}{A} \; .$$

Half the material N_0 will be left at time τ , one-quarter will be left at time 2τ , etc. When will it all be gone? We see from (1) that in order for N to approach 0 , t must approach infinity (and this is why the whole-life is a useless measure).

Let us consider a specific example. The half-life of radium is about 1600 years, and the corresponding decay coefficient A is

$$A \approx \frac{0.693}{1600} \approx 0.000433 \text{ per year.}$$

If we start with some given amount (N_0) and wait a hundred years, we get $\log \frac{N}{N_0} \approx -0.0433$, and consequently $N \approx 0.958 N_0$ is the amount left. Thus only 4.2% disappears in one hundred years.

The basis for applying (1) to radioactivity is statistical, i.e., it holds in the sense of an average. Although the physical process is governed by probability, and we cannot tell when any one atom will disintegrate, it is quite useful to determine the mean life-time per atom. We start with N_0

atoms at $t = 0$ and end up with 0 atoms as t approaches infinity, and we are interested in the average length of time that an atom exists.

If n_1 atoms disappear at time t_1, n_2 atoms at time t_2, etc., where

$$t_1 < t_2 < \dots < t_k$$

and $\sum_{i=1}^{k} n_i = N_0$, then the mean life-time of an atom is the average value

$$T = \frac{1}{N_0} \sum_{i=1}^{k} n_i t_i \text{ .}$$

If the total number of atoms present in the interval (t_{i-1}, t_i) is N_i, then N_{i+1} is the number present in the interval (t_i, t_{i+1}), and

$$n_i = N_i - N_{i+1} \text{ .}$$

We then have

$$T = \frac{1}{N_0} \sum_{i=1}^{k} (N_i - N_{i+1}) \text{ .}$$

Now, as we did in setting down the differential Equation (1), we blur the conception of radioactive decay as the result of individual instantaneous disappearances and treat the relation between t and N as though t were given by a continuous function of a real variable N, although, in fact, N is a discrete variable and the relation is not a function.* If t were given by a continuous function of N we could regard the preceding sum as a Riemann sum (with the partition points N_i given in the reverse of the usual order). Thus in our continuous model we define the mean lifetime as

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*If we consider N as a function of t and take a microscopic approach in which the disintegrations of individual atoms are observed, then N is piecewise constant and jumps discontinuously at time t_i from N_i to N_{i+1}. If we take a microscopic approach in which the amount of substance is measured in grams, or even in millimicrograms, the disintegration of single atoms is insignificant and the difference between the true function $t \longrightarrow N(t)$ and the continuous decreasing function $t \longrightarrow N_0 e^{-At}$ is ignored. Thus, we take here a point of view which should be considered in the light of the example of Section 2-1. In Section 2-1 we pointed out that a poor choice of scale may conceal features which concern us. Here we observe in contrast that a proper choice of scale may help us disregard small features which distract and may permit us to treat by simple analysis a problem which might otherwise be extremely difficult. For many applications, a suitable choice of scale is vital.

$$T = -\frac{1}{N_0}\int_{N_0}^{0} t\,dN\,. \tag{3}$$

Now we can express the mean lifetime T in terms of the decay coefficient A. From (1) we have

$$t = -\frac{1}{A}\log\frac{N}{N_0}\,. \tag{4}$$

Using this expression for t in (3) we may integrate (Exercises 9-3, No. 14, also see Example 10-2f and Example 10-4a) to obtain

$$T = \frac{1}{A}\,. \tag{5}$$

Thus the mean lifetime is the reciprocal of the decay coefficient.

We have a model for simple radioactive decay. What is left after an atom disintegrates? Many things, including "daughter" atoms which can also disintegrate. Later we discuss the decay of the daughter population as well.

The simple decay model we have been considering also describes essential features of many other phenomena. For example, within the same mathematical structure we need change only the names of the characters in order for the results to apply to the molecules of air in your lungs. Suppose that N_0 is the total number of molecules present, $N(t)$ is the number that have not had collisions by time t, and that the mean time between collisions is T. Then $\frac{N(t)}{N_0}$, the probability that any one molecule goes for time t without a collision, follows directly from (1) and (5):

$$\frac{N(t)}{N_0} = e^{-t/T}\,. \tag{6}$$

We have the same equation as before, but the symbols now play different physical roles, and of course the over-all plot is quite different. Were we to continue the present story we would require much additional structure: if the mean velocity of the molecules is v, then $L = Tv$ is called the mean free path--the average distance a molecule travels between collisions--a concept basic to statistical mechanics, the theory which bases the physical properties of matter on the motion of molecules.

(ii) Attenuation with distance. The model for a process that decays with time can also be applied to processes that attenuate (weaken) with distance. Let us relabel the variable t of Section 9-3 (i), call it "distance," and write it as "x" to keep the new role in view. We thus have

$$(1) \qquad \frac{dN(x)}{dx} = -AN(x)\ ,\ N(0) = N_0\ ;\ \text{hence}\ \ N(x) = N_0e^{-Ax}$$

which we call the attenuation equation.

The alternation of the earth's atmosphere with altitude is approximately described by (1), where x is the height above the earth's surface, $N(x)$ the number of molecules per unit volume (the density) of atmosphere at height x, and N_0 the density at ground level. The atmosphere contains different kinds of molecules with different masses m, so that m should be introduced as a physical parameter and we should write

$$(2) \qquad N(m,x) = N_0(m)e^{-A(m)x}$$

for each species of molecule. Under more detailed study it turns out that $A = ma$ where a does not depend on m (but depends on temperature and the acceleration of gravity).

As an alternative setting for (1), visualize a narrow beam of particles incident on the face of a medium of more massive particles as in Figure 9-3.

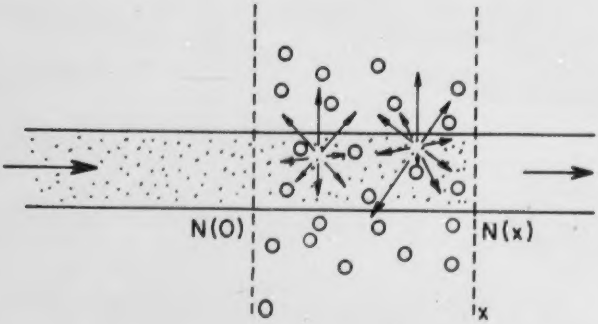

Figure 9-3

There are N_0 particles per unit volume of the incident beam and nothing happens to them until they encounter the medium that starts at $x = 0$. Then as the beam penetrates, its lighter particles hit the heavier ones of the medium and go off in other directions: particles of the incident beam are lost to other directions by scattering. Thus $N(x)$, the density of particles in the incident beam at a distance x within the medium is less than N_0; this attenuation is governed by the scattering coefficient per unit length A. (For other processes, density along the beam may attenuate because its particles combine with the heavier ones of the medium.)

The principal characters in the above are particles--a loose characterization than can stand for electrons, protons, etc. If we now relabel N as energy density per unit volume or intensity, then the same plot also holds for light-rays, X-rays, γ-rays, and all other kinds of waves meeting appropriate obstacles. For the particles, we mentioned one physical parameter, the mass m, and spoke of some particles being scattered by those of more mass. For waves, the appropriate physical parameter is the wavelength λ, e.g., for equally spaced ripples on a lake the wavelength is the distance between successive crests. The longest wavelengths associated with visible light give the sensory impression called "red" and they are about twice as long as the shortest of the wavelengths associated with visible "blue." From blue light to ultra-violet to x-rays to γ-rays we go to shorter and shorter electromagnetic wavelengths; from red light to infrared to microwaves to radio waves we go to longer electromagnetic wavelengths. The wave picture is not confined to electromagnetic effects, we can also talk about sound waves, water waves, and even the waves of "probability amplitude" associated with electrons, neutrons, and other fundamental particles.

With N for intensity, (1) in terms of an appropriate A describes the attenuation of a beam of sunlight penetrating a cloud or a layer of fog, etc. We can use (1) to determine the thickness of lead shields to be used with medical x-ray equipment or with a nuclear reactor to reduce stray radiation to a tolerable value. We could discuss any of the above in greater detail but instead let us talk about something more colorful.

Let us consider Rayleigh's theory for the color of the sky. The essential feature of sunlight is that it is made up of light of different colors from red to blue (the visible spectrum) with associated wavelengths λ_r to λ_b where (approximately)

$$\lambda_r = 2\lambda_b . \tag{3}$$

The wavelength λ of an intermediate color (orange, yellow, green) satisfies $\lambda_r > \lambda > \lambda_b$. Rayleigh showed that when a beam of light of wavelength λ is scattered by the molecules of the earth's atmosphere (mainly nitrogen and oxygen), the intensity $N(\lambda,x)$ along the beam is governed approximately by (1) with

$$A(\lambda) = \frac{C}{\lambda^4} \tag{4}$$

where C is independent of λ. (In the chapter on optics and waves we discuss this in detail.)

From (4) and (3) we have

$$\frac{A(\lambda_b)}{A(\lambda_r)} = \frac{\lambda_r^4}{\lambda_b^4} = \frac{(2\lambda_b)^4}{\lambda_b^4} = 16 \tag{5}$$

and consequently

$$\frac{N(\lambda_b,x)}{N_0(\lambda_b)} = e^{-A(\lambda_b)x} = e^{-16A(\lambda_r)x} = \left[\frac{N(\lambda_r,x)}{N_0(\lambda_r)}\right]^{16} . \tag{6}$$

Thus the blue component of white light is 16 times more strongly attenuated than the red. A beam of white sunlight reddens with penetration into the earth's atmosphere because it loses its blue component more rapidly than its red. The blue that is lost from the sunbeams by scattering gives the sky its blue color in directions away from the sun. The direct beams from the overhead sun are still relatively white because they have not lost that much blue. The reddening of the direct beams is best seen when the sun is low on the horizon and its rays traverse maximum distance through the scattering atmosphere; the clouds in the path of these rays are bathed in red. Such colored effects and other scattering phenomena arising from water drops, dust particles, and other impurities in the atmosphere are more fully discussed in the deeper researches of the poets.

(iii) Mother-daughter reactions. As mentioned previously, when a radioactive atom (the mother atom) disintegrates it may give rise to a daughter atom which can also disintegrate. Let us now consider such mother-daughter relations.

Suppose we have N_1 mother atoms which decay at the rate

$$\frac{dN_1}{dt} = -A_1N_1 \,, \qquad N_1(0) = N_{10} \,. \tag{1}$$

The rate at which the mothers decay equals the rate at which the daughters are created. But the daughters also decay on their own. If N_1 mothers with birth coefficient A_1 give rise to N_2 daughters with decay coefficient A_2, the rate of change of the number of daughters is given by

$$\frac{dN_2}{dt} = A_1N_1 - A_2N_2 \,, \qquad N_2(0) = 0 \,. \tag{2}$$

Equations (1) and (2) form a pair of simultaneous equations for determining N_1 and N_2.

Let us first consider a limiting case such that the mothers decay very slowly compared to the daughters, that is A_1 is very much smaller than A_2. This corresponds, for example, to the behavior for the pair radium-radon. For radium mothers, the half-life is approximately 1600 years: $A_1 \approx \frac{1}{1600}$ per year. The radon daughters have a half-life of about 4 days: $A_2 \approx \frac{1}{4}$ per day $\approx 360 \cdot \frac{1}{4}$ per year = 90 per year. Thus $A_2 \approx 144{,}000A_1$ and we may hold the number N_1 of mothers constant for the purpose of obtaining a first approximation of $N_2(t)$.

Equation (2), with N_1 constant, has the general form which appears repeatedly in this chapter:

$$\frac{dy}{dx} = g(y) \,.$$

This can easily be recast as an equation for the inverse function:

$$\frac{dx}{dy} = \frac{1}{g(y)} \,.$$

Applying the Fundamental Theorem we obtain

$$x = C + \int_a^y \frac{1}{g(u)}\, du \,,$$

where a and y lie in any interval where $\frac{1}{g(u)}$ is continuous. This method will be applied throughout the chapter (see Section 10-9).

Thus we regroup the terms in (2), replace t by u, and integrate from $t_1 = 0$ to $t_1 = t$.

$$-A_2 \int_0^t \frac{dN_2}{A_1N_1 - A_2N_2} = -A_2 \int_0^t du ,$$

subject to the approximation that N_1 is constant. We obtain

$$\log K(A_1N_1 - A_2N_2) = -A_2t ,$$

whence

$$K(A_1N_1 - A_2N_2) = e^{-A_2t} ,$$

where the integration constant K is to be determined from the initial conditions at $t = 0$. There are no daughters at $t = 0$:

$$K(A_1N_1 - 0) = e^{-A_2t}\Big|_{t=0} = 1 ,$$

whence

$$K = \frac{1}{A_1N_1} .$$

Consequently, the number of daughters at time t is given approximately by

$$N_2 = \left(\frac{A_1}{A_2}\right) N_1(1 - e^{-A_2t}) . \tag{3}$$

If A_2t is very large, then N_2 approximates $\frac{A_1N_1}{A_2}$, i.e., the number of daughter atoms approaches a fixed fraction of the relatively inert mother substance. (This is called long term or secular equilibrium.) What does this mean? It corresponds, for example, to the case where N_2 is a gas (such as radon) in a closed container, and a situation where just as much N_2 is created (from N_1) as is destroyed by radioactive decay. The birth rate of N_2 equals its death rate, so that $\frac{dN_2}{dt}$ is zero; our result as t approaches infinity in (3) is thus the same as that obtained by equating (2) to zero. Equilibrium corresponds to

$$\frac{dN_2}{dt} = 0 \ ; \ N_2 = \left(\frac{A_1}{A_2}\right)N_1 . \tag{4}$$

Now we take into account the decay of the mother population in the original problem. We substitute the solution of (1), i.e., $N_1 = N_{10}e^{-A_1t}$ into (2) to obtain

$$\frac{dN_2}{dt} + A_2N_2 = A_1N_{10}e^{-A_1t} . \tag{5}$$

The solution of (5) for N_2 is left as an exercise (Exercises 9-3, No. 17). Taking $N_2(0) = 0$, we have

$$N_2 = \frac{A_1}{A_2 - A_1} N_{10}(e^{-A_1t} - e^{-A_2t}) . \tag{6}$$

which reduces to (3) if A_1 is very much less than A_2 , and $A_1t = 0$. In distinction to the approximation (3), the present complete form N_2 of (6) vanishes both for $t = 0$ and as t approaches infinity; consequently N_2 must have a maximum at a specific value of t .

If we differentiate (6) with respect to t we obtain

$$\frac{dN_2}{dt} = \frac{A_1N_{10}}{A_1 - A_2} (-A_1e^{-A_1t} + A_2e^{-A_2t}) . \tag{7}$$

This vanishes, and N_2 has a maximum, when

$$A_1e^{-A_1t} = A_2e^{-A_2t} , \frac{A_1}{A_2} = e^{(A_1 - A_2)t} .$$

From the logarithmic form, we obtain

$$t = \frac{\log A_1 - \log A_2}{A_1 - A_2} \tag{8}$$

as the time when the number of daughters is largest. The maximum number of daughters is $N_2 = N_{10}e^{-A_2t} = \left(\frac{A_1}{A_2}\right)N_{10}e^{-A_1t} = \left(\frac{A_1}{A_2}\right)N_1$ as in (4).

(iv) Biology. The basic neural process is the excitation and propagation of nerve impulses initiated by a stimulus. One way of studying this process is to excite the nerve fibers by an electrical stimulus V (the voltage associated with a direct current, the discharge of a condenser, or an alternating current), and to measure the characteristic effects. The voltage V must be greater than a threshold value V_e , the minimum value of V that is just sufficient to cause the effects. A simple model describes the onset of the effects in terms of a local latency $N(t)$ (also called the "excitatory function") such that

(1) $$\frac{dN(t)}{dt} = KV(t) - AN(t) ,$$

where K is the growth of the latency per second per unit stimulus, and A is its decay coefficient. Thus the growth of N increases with the magnitude of the stimulus and decreases with N. (The function N may represent the difference between the concentration of an exciting ion at an electrode while V is applied and its concentration for $V = 0$.) If $N(t)$ reaches (or exceeds) a threshold value N_e, then the nerve becomes excited (and a characteristic physical-chemical wave with an associated electric potential propagates along the fiber).

The simplest application of (1) is to the situation

(2) $$N(0) = 0 , \quad V(t) = V = \text{constant}$$

which corresponds to the application of a constant stimulus at time $t = 0$. A comparison with Equations (2) and (3) of Section 9-3(iii) yields the solution of (1) and (2):

(3) $$N = \frac{KV}{A}(1 - e^{-At}) .$$

Thus as t approaches infinity, we see that N approaches its largest value $N_{max} = \frac{KV}{A}$. Consequently excitation will occur if

(4) $$N_{max} = \frac{KV}{A} \geq N_e$$

or equivalently if the stimulating voltage satisfies

(5) $$V \geq \frac{AN_e}{K} = V_e ,$$

where V_e is the threshold stimulus mentioned previously. (The value V_e is known as the rheobase, the threshold or liminal value of the constant voltage necessary for excitation.)

Assuming that $V > V_e$ (so that excitation must occur), then the nerve becomes excited at the time t_e when the value N in (3) reaches the threshold value:

(6) $$N_e = \frac{KV}{A}(1 - e^{-At_e}) ,$$

or equivalently,

(7) $$t_e = \frac{1}{A} \log \frac{V}{V - V_e} ,$$

which is the latent period that elapses between the establishment of the constant stimulus and the release of excitation. If $V < V_e$, no value of t_e exists. If $V = V_e$, then the latent period t_e approaches infinity; however, this is an inconvenient length of time for measurement. A more convenient measure is the value of t_e corresponding to $V = 2V_e$:

(8) $$\tau = \frac{\log 2}{A} \approx \frac{0.693}{A} .$$

This is known as the chronaxie τ --the latent time before excitation for the case of a stimulating voltage equal to twice the threshold value.

What have we been doing in the above? Essentially we have changed the names of the concepts introduced for radioactive decay end showed that much of biology, physics and chemistry involves the same simple ideas. Let us now generalize the mathematical development to nonconstant values of V in (1).

If V is a function of time, we solve Equation (1) in terms of $V = V(t)$ by proceeding essentially as for Equation (5) of Section 9-3(iii) (see No. 17) to obtain:

(9) $$N = e^{-At} \left\{ N_0 + K \int_0^t V(t_1) e^{At_1} dt_1 \right\} .$$

If $N(0) = N_0 = 0$, and V is a constant, then (9) reduces to (3).

If we stimulate the process by discharging a condenser of initial charge q , capacity C , through a resistance R , then (see Exercises 9-3, No. 8(a))

(10) $$V(t) = (\frac{q}{C}) e^{-t/CR} .$$

Substituting in (9) and integrating, we obtain, for $N_0 = 0$,

(11) $$N = \left[\frac{KqR}{(CRA - 1)} \right] [e^{-t/CR} - e^{-At}]$$

which is simply Equation (6) of Section 9-3(iii) with different labels. Thus the excitation function N has a maximum when

(12) $$t = \left[\frac{CR}{1 - CRA} \right] \log(\frac{1}{CRA}) .$$

If the maximum value of N is precisely the threshold value, then t of (12) is the corresponding latent time from onset of stimulus V to release of a wave of activity in the nerve. The corresponding initial voltage

$V(0) = \frac{q}{C}$ is the threshold initial voltage of the condenser for excitation to occur. If the maximum does not equal the threshold, then we relate the condenser's characteristics to the threshold by equating N in (11) to N_e in (6) and using $V_e = \frac{AN_e}{K}$ to eliminate K .

If we stimulate the process by a sinusoidal alternating current, then the applied voltage is

$$V(t) = V_0 \sin \omega t \ , \tag{13}$$

where V_0 is the constant amplitude. Substituting in (9) for $N(0) = 0$ we have

$$N(t) = e^{-At} KV \int_0^t e^{At_1} \sin \omega t_1 dt_1 \ . \tag{14}$$

We shall learn how to handle the new integral in Chapter 10 (also see Chapter 8, Miscellaneous Exercises, No. 19b); here we simply quote the result:

$$\int_0^t e^{At} \sin \omega t dt = \frac{-\omega e^{At} \cos \omega t + Ae^{At} \sin \omega t}{\omega^2 + A^2} \ ;$$

this formula may be checked by differentiation. Consequently the solution of (14) is

$$N(t) = \frac{KV_0}{\omega^2 + A^2} (A \sin \omega t - \omega \cos \omega t + \omega e^{-At}) \ . \tag{15}$$

The exponential term of (15) is significant only for small values of t . As t increases, e^{-At} becomes negligible and (15) reduces to

$$N(t) \approx \frac{KV_0}{\omega^2 + A^2} (A \sin \omega t - \omega \cos \omega t) \ . \tag{16}$$

This periodic approximation has equally spaced extrema in time, which occur when

$$\frac{dN}{dt} = \frac{KV_0 \omega}{\omega^2 + A^2} (A \cos \omega t + \omega \sin \omega t) = 0 \ , \ t = \frac{1}{\omega} \tan^{-1} \left(\frac{-A}{\omega}\right) \ . \tag{17}$$

Substituting these values of t into (16), we find that the maxima of N equal

$$N_{max} = \frac{KV_0}{(\omega^2 + A^2)^{1/2}} \ . \tag{18}$$

If we equate N_{max} to the threshold value N_e, then V_0 of (18) corresponds to the threshold value of the amplitude of the sinusoidal stimulus, say V_{0e}; however, the proportion between V_{0e}^2 and $\omega^2 + A^2$ has very limited validity in nature.

These examples cover most of the modes of stimulation which are likely to be used in the laboratory.

9-4. Bounded Growth. Competition.

(i) A more realistic story for the spread of stories. Let us return to our model for the spreading of stories (or of diseases, or of ink blots), and introduce more structure. Previously we assumed that the rate of change of the number of people who knew the story at time t was proportional only to the number itself:

$$(1) \qquad \frac{dN(t)}{dt} = A\,N(t)\;, \qquad N(0) = N_0\;.$$

This is all right as far as it goes, but it ignores the fact that there is an upper bound (say K) on the number available to hear it: there are finitely many people on earth, some don't talk your language, some don't talk at all, and some never listen. Furthermore, although we may tell the same person the same story a dozen times, each listener should be counted only once.

In view of these considerations, we replace (1) by

$$(2) \qquad \frac{dN}{dt} = AN\frac{(K - N)}{K}\;, \qquad N(0) = N_0\;,$$

where $N = N(t)$ know the story at time t and are available to spread it, and $K - N$ do not know the story, have good hearing, and are enthusiastic listeners and potential gossips. The factor $\frac{(K - N)}{K}$ is the fraction of the population available for the further spread of the story. Dividing both sides of (2) by K, we introduce $r = \frac{N}{K}$ as the fraction of the available population that know the story, and work with the conditions

$$(3) \qquad \frac{dr}{dt} = Ar(1 - r)\;, \quad r_0 = \frac{N_0}{K}\;,$$

where r_0 is the fraction at $t = 0$. Our original model (1) yielded an unbounded increase in N as t approaches infinity. What does the present model give? We expect that $\lim\limits_{t\to\infty} r = 1$, that is, that eventually everyone knows the story. (Even this model is far from complete, but at least this kind of result is acceptable.) From (3) we see that $\frac{dr}{dt}$ approaches 0 as r approaches 1, i.e., that r stops changing when everyone knows the story. From the discussion for Equation (4) of Section 9-3(iii) we may surmise that $\frac{dr}{dt}$ approaches 0 as t approaches ∞, but let us solve (3) and see the details.

From (3), we write $\int \frac{dr_1}{r_1(1 - r_1)} = \int A\,dt_1$, where r_1 and t_1 are dummy variables. Since

$$\frac{1}{r(1 - r)} = \frac{1}{r} + \frac{1}{1 - r}\;,$$

(for such decompositions of fractions into partial fractions see Section 10-5), we have

$$\int\left[\frac{1}{r_1} + \frac{1}{1 - r_1}\right] dr_1 = \log \frac{r_1}{1 - r_1}\Bigg|_{r_0}^{r} = \log \frac{r}{1 - r} - \log \frac{r_0}{1 - r_0} = At \ .$$

Solving for r, we obtain

$$r = \frac{r_0 e^{At}}{1 + r_0(e^{At} - 1)} \ . \tag{4}$$

If t is small, then the denominator approximates unity and $r \approx r_0 e^{At}$, in accord with the simplest model (1). On the other hand if t is large, we rewrite (4) as

$$r = \frac{r_0}{r_0 + (1 - r_0)e^{-At}} \tag{5}$$

from which we see that r approaches 1 as t approaches ∞.

The above model indicates some of the essentials but it is still incomplete. However, it is good enough to show that although you may still have not heard the story about Al (see Appendix 9), you should by now have heard about Helen of Troy.

(ii) Growth and competition. A more general equation, which includes as special cases Equations (1) and (3) of Section 9-4(i), is

$$\frac{dN}{dt} = AN - BN^2 \ , \qquad N(0) = N_0 \ . \tag{1}$$

This is called the logistics equation. We still call A the growth coefficient, and we may call B the braking coefficient because the term $-BN^2$ slows the growth. The equation of unregulated growth, $\frac{dN}{dt} = AN_1$, permits N to increase beyond any bound as t approaches ∞; Equation (1) does not. What bound does (1) impose on N? We see that $\frac{dN}{dt} = 0$ when $A = BN$ as in Section 9-4(i): the corresponding value $N = \frac{A}{B}$ must be the equilibrium value which N approaches as t approaches infinity.

A model for the growth of populations of different countries used at one time was essentially $\frac{dN}{dt} = AN$, and this led to dire predictions as to the fate of mankind (Malthus). Then a more refined model, essentially (1), was introduced (by Verhulst) and appropriate A's and B's for various countries were obtained from their earlier census records; the projected growth curves were remarkably accurate (at least for all countries except Verhulst's--Belgium). The build-up of population growth arising from A has been interpreted as due to cooperation between people, and the slow-down associated with B as due to competition between people for limited resources. The competition assumption can be made quite plausible: if p is the probability that a person wants a particular thing, then p^2 is the probability that two persons want it simultaneously; if there are N persons, since there are $\frac{N(N-1)}{2}$ possible competing pairs then the total probability of competition $p^2 \frac{N(N-1)}{2}$ is approximately proportional to N^2 which then becomes a plausible measure of the simultaneous desire or of the competitive urge.* However, the reason for regarding AN as a measure of cooperation is not clear. A probability interpretation similar to that for BN^2 indicates that AN corresponds to N persons acting quite independently of each other; this may well be as close to cooperation as one can expect from a group, and SMSG authors have therefore taken this as the guiding principle for preparing their textbooks.

Let us solve (1) by the same procedure we used for (3) of Section 9-4(i). The steps are essentially the same, and we get

$$\log \frac{N}{1 - CN} - \log \frac{N_0}{1 - CN_0} = At \ , \ C = \frac{B}{A}$$

and consequently

$$N = \frac{N_0 e^{At}}{1 + N_0(e^{At} - 1)\frac{B}{A}} \ . \tag{2}$$

If t is small, then the denominator is approximately $1 + N_0 tB \approx e^{N_0 tB}$, and (2) reduces to

$$N \approx N_0 e^{t(A - BN_0)} = N_0 e^{t(A-D)} \ , \tag{3}$$

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*Note that the linear term can be subsumed in the growth term of the differential equation.

where $D = BN_0$ is introduced as an abbreviation. Thus for small t, the result has the same form as for the simple model in terms of the growth coefficient $A - D$. On the other hand

(4) $$\lim_{t\to\infty} N = \frac{A}{B} = \frac{AN_0}{D}$$

in accord with our guess that $N = \frac{A}{B}$ must represent the long-term equilibrium of the population.

(iii) Forgetting and learning. The previous sections also provide a simple model for forgetting and learning, at least of unconnected chains of nonsense syllables invented by psychologists for test purposes. Thus (as proposed by Von Foerster*) we consider

(1) $$\frac{dN}{dt} = -AN + BN(N_0 - N), \qquad N(0) = N_0,$$

where N_0 is the initial number of items memorized (dates, telephone numbers, unconnected theorems, etc.), A is a forgetting coefficient, and BN_0 is a memorization coefficient. The notion behind (1) is that your head is originally filled with N_0 "carriers" of information; some carriers (AN) just lose their information forever; some (BN) lose information in the sense that they pass information on to the empty $N_0 - N$ carriers.

Integrating (1) (the present (1) is the same as Equation (1) of Section 9-4(ii) with a new growth coefficient $BN_0 - A$), we write the solution of (1) as

(2) $$\frac{N}{N_0} = \frac{D - A}{D - Ae^{-(D-A)t}}, \quad D = N_0B.$$

The remembrance $R = \lim_{t\to\infty} \frac{N}{N_0}$ (as defined by Von Foerster) depends critically on the magnitude of $\frac{D}{A}$. If $D > A$, then

(3) $$R = \frac{D - A}{D} = 1 - \frac{1}{\frac{D}{A}}, \qquad (\frac{D}{A} > 1).$$

On the other hand, if $D \leq A$, then

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*H. Von Foerster, "Quantum Theory of Memory," Transactions of Sixth Conference on Cybernetics, 1950, pp. 112-134.

(4) $$R = 0 , \qquad (\frac{D}{A} \leq 1) .$$

Thus the remembrance of things past is zero for $\frac{D}{A} \leq 1$, and then increases towards unity as $\frac{D}{A}$ increases from unity.

(iv) Chemical reactions. Suppose we have a chemical substance with initial concentration C (gram-molecules per unit volume) which is reacting in time with something unimportant and plentiful to form another substance with concentration N . The rate of change of N is proportional to the concentration of the original substance at time t , that is, to $C - N$:

(1) $$\frac{dN}{dt} = A(C - N) , N_0 = 0 ,$$

where N is the concentration of the new substance and A is called the reaction rate. Equation (1), which is known as the law of mass action, is essentially the special case of (2) in Section 9-3(iii) for A_1 much smaller than A_2 : by inspection of (3) in Section 9-3(iii) the solution (i.e., the concentration of the solution) is

(2) $$N = C(1 - e^{-At}) .$$

Equivalently, Equation (1) is a shifted version of the simplest decay equation; setting $M = C - N$ in (1), we obtain

$$\frac{dM}{dt} = -AM , M_0 = C - N_0 = C ,$$

which is the same as Equation (1) of Section 9-3(i) and leads directly to (2) for $N = C - M$.

From (2), we see that if $t = 0$, then $N = 0$. Further, $\lim_{t \to \infty} N = C$ so that all of the original substance eventually reacts. We may isolate A in the form

(3) $$A = \frac{1}{t} \log \frac{C}{C - N(t)}$$

which you may well use in a later chemistry course to determine A by measuring C , N , and t .

In a bimolecular reaction, we have two different substances with initial concentrations C_1 and C_2 which react at a rate determined by A to produce a third substance whose concentration is N :

(4) $$\frac{dN}{dt} = A(C_1 - N)(C_2 - N) , \qquad N_0 = 0 .$$

This is just another variation of the logistics equation (Equation (1) in Section 9-4(ii)), and the solution can be obtained from the previous ones. However, to emphasize this basic integration procedure, we again integrate using a decomposition into partial fractions:

$$\int \frac{dN}{(C_1 - N)(C_2 - N)} = \frac{1}{C_1 - C_2} \int \left| \frac{1}{C_2 - N} - \frac{1}{C_1 - N} \right| dN = \frac{1}{C_1 - C_2} \log \frac{C_1 - N}{C_2 - N} = At + K ,$$

where the integration constant K is obtained from the condition $N = 0$ at $t = 0$:

$$K = \frac{\log \frac{C_1}{C_2}}{C_1 - C_2} .$$

Thus

$$A = \frac{1}{t(C_1 - C_2)} \log \frac{C_2(C_1 - N)}{C_1(C_2 - N)} , \tag{5}$$

and

$$N = C_1 \frac{1 - e^{(C_1 - C_2)At}}{1 - \left(\frac{C_1}{C_2}\right) e^{(C_1 - C_2)At}} . \tag{6}$$

The case $C_1 = C_2 = C$ may be obtained from the limit of (6) as C_1 approaches C_2 . Equivalently, we start with

$$\frac{dN}{dt} = A(C - N)^2 , \qquad N_0 = 0 , \tag{7}$$

and integrate:

$$\int \frac{dN}{(C - N)^2} = \frac{1}{C - N} + K = At .$$

The constant equals $K = \frac{-1}{C}$, and therefore

$$A = \frac{1}{t} \frac{N}{C(C - N)} , \quad N = \frac{C^2 At}{1 + CAt} . \tag{8}$$

The equation for opposing unimolecular and bimolecular reactions has the form

$$\frac{dN}{dt} = A(C - N) - BN^2 , \qquad N_0 = 0 . \tag{9}$$

We do not discuss this case but merely reduce it to a previous form. Thus we introduce

(10) $$D_1 = -\frac{1 + K}{\frac{2B}{A}} , \quad D_2 = -\frac{1 - K}{\frac{2B}{A}} , \quad K = \sqrt{1 + \frac{4CB}{A}}$$

in order to rewrite (9) as

(11) $$\frac{dN}{dt} = -B(D_1 - N)(D_2 - N) .$$

We now have the form (4) with the previous A , C_1 , C_2 replaced by $-B$, D_1 , D_2 , and the corresponding results may be written down by inspection.

We could go on to higher-order reactions of the form

(12) $$\frac{dN}{dt} = A(C_1 - N)(C_2 - N)(C_3 - N) \ldots ,$$

(Exercises 9-5, Nos. 1-3), but we must finish the story.

9-5. Conclusion.

(i) Sociology. Now we could rehash everything. We could change the names of the symbols in the previous equations and talk about profound sociological problems. Instead we introduce a more general model for the growth of populations, one which includes practically all of our previous equations as special cases, and scarcely talk at all.

In Section 9-4(ii), the growth of a population of N individuals was described by

$$\frac{dN}{dt} = AN - BN^2 , \tag{1}$$

where A is the growth coefficient, and B is the braking coefficient. Let us now introduce more structure. We may write $A = \alpha - \beta$, where α is the birth coefficient (the birth rate per individual) and where β is one of two death coefficients. If we assume that the population is confined to an area S, then we may write the other death coefficient as $\frac{C}{S}$, i.e., the death rate per individual $\frac{CN}{S}$, increases as S decreases or as N increases (no room to live). Thus the total death rate is $(\beta + \frac{CN}{S})N$. Using $\gamma = \frac{C}{S}$ instead of B (merely for esthetic reasons) we rewrite (1) as

$$\frac{dN}{dt} = AN - (\beta + \gamma N)N . \tag{2}$$

A more general model (considered by Rashevsky[*]) is that for the growth of a population consisting of two types of individuals with different birth and death characteristics. The total population is

$$N = N_1 + N_2 , \tag{3}$$

and N_1 and N_2 are specified by the simultaneous equations

$$\begin{aligned} \frac{dN_1}{dt} &= \alpha_{11}N_1 + \alpha_{12}N_2 - [\beta_1 + \gamma_1(N_1 + N_2)]N_1 , \\ \frac{dN_2}{dt} &= \alpha_{21}N_1 + \alpha_{22}N_2 - [\beta_2 + \gamma_2(N_1 + N_2)]N_2 , \end{aligned} \tag{4}$$

where the α's, β's, and γ's are all constants. The terms proportional to α represent the contributions of the two groups to the birth rates; the death rates that depend on γ_i (with $i = 1$ or 2) depend not only on N_i, but also on the total population $N_1 + N_2 = N$. The system of Equations (4) generalizes practically all the other equations considered previously in this chapter.

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[*] N. Rashevsky, Mathematical Theory of Human Relations, Principia Press, Indiana, 1947.

We do nothing with (4) in the text, but as an exercise obtain all the previous equations that we considered and that can be obtained from (4) under suitable restrictions. Talk about active individuals and passive individuals; talk about active and passive disobedience; talk about social aggregates, freedom, crime, war, propaganda, etc. Write a book about it; call it 'War and Peace.'

(ii) Coda. We make observations, we create models, we make predictions; we make more observations, more models, more predictions; we day-dream and jump to conclusions; we seek to verify our guesses, and keep the very few that pass the tests. By such means, by a mixture of measurements, mathematics, and mysticism we seek to "understand" what is going on around us. If we can predict and describe a process and relate it to analogous processes that we know about, we are content--for a while.

As we apply mathematics to the various sciences, we soon discover that at a fundamental level there appear to be only a few different kinds of processes going on. The equations are the same, only the names of the functions and variables change from science to science. The stages and settings are very different, and the over-all plots vary; but the sub-plots are routine, the actors go through the same motions, and only the names of the characters are changed.

Exercises 9-2

1. A colony of bacteria grows at a rate proportional to its population. Initially there are 2,000 bacteria and one day later, 5,000 . What should the population of the colony be after 5 days if it grows unrestrictedly?

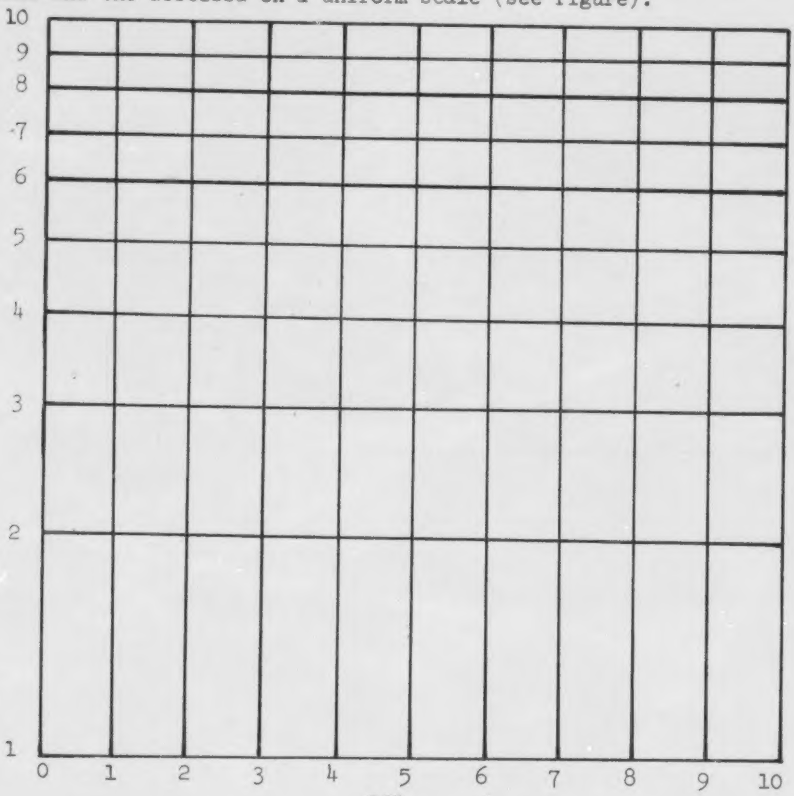
2. The rate of production of a given chemical in a given reaction increases by 2% for each degree of increase in temperature. What increase in temperature is needed to double the production rate?

3. Show for any solution of Equation (3) that over a fixed time period τ the change in N is in constant proportion to the value of N at the beginning of the period, independently of the initial time t .

4. Write the differential equation for the growth of a bacterial population which increases 2.5% every hour. If there are N_0 bacteria at the start, how many are there at the end of 10 hours?

5. The population of a city has been growing at a rate proportional to itself. If the population is now 40,000 and 25 years ago it was 15,000 , find the anticipated population 10 years hence.

6. Semi-logarithmic coordinates represent the ordinate on a logarithmic scale and the abscissa on a uniform scale (see figure).



(a) The graph of $y = Ae^{bx}$ is given by a straight line on semilogarithmic paper. Why?

(b) The census figures for the total population of the United States obtained at ten-year intervals from 1790 to 1960 are given in millions as follows: 3.93 , 5.31 , 7.24 , 9.64 , 12.9 , 17.1 , 23.2 , 31.4 , 38.6 , 50.2 , 62.9 , 76.0 , 92.0 , 106 , 123 , 132 , 151 , 179 .

Plot the population vs. time on semilogarithmic graph paper. Over what time intervals do the points appear to lie on a straight line? Use Formula (1) to obtain a reasonable average value of b for each of these periods.

(c) Do the same for the census figures for the state of California for the period 1850 to 1960 (figures in hundreds of thousands): 0.926 , 3.80 , 5.60 , 8.65 , 12.1 , 14.9 , 23.8 , 34.3 , 56.8 , 69.1 , 106 , 157.

(d) Look up the figures for your own state in an almanac and study its population growth in the same fashion.

Exercises 9-3

1. If the half-life τ of a radioactive substance is given in seconds, show that the fraction of the substance decaying in one second is approximately $\frac{\log 2}{\tau}$. (Hint: Assume that τ is a large number.)

2. Verify that the half-life of a radioactive substance is independent of the initial time and the initial amount of the substance.

3. A 5000 cubic foot garage containing a high concentration of carbon monoxide is being flushed out by an air pump whose capacity is 1000 ft^3/min. Assuming that the air mixture in the garage remains uniform (perfect mixing), determine how long it takes for the concentration of carbon monoxide to fall to $\frac{1}{10}$ its initial value.

4. For a small body in air or liquid, the rate of heat loss is approximately proportional to the difference in temperature between the body and the surrounding medium.

   A thermometer which registers 72° indoors is taken outdoors where the temperature is 12° . One minute later the thermometer registers 42° .

   (a) Obtain a formula for the thermometer reading r at any time t .

   (b) What is the reading at the end of two minutes?

(c) If the thermometer is left outdoors how long does it take to reach a reading of 18° ?

(d) On another day, it takes one minute out-of-doors for the thermometer to drop from its indoor reading of 72° to a reading of 32° . Since it is too cold to stay outside and wait for the thermometer to reach an equilibrium reading, calculate the outside temperature.

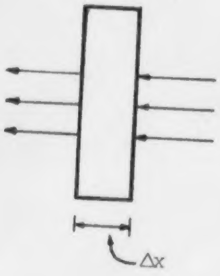
5. A thermometer which registers 70° indoors is taken outdoors. Five minutes later it registers 65° and ten minutes after it was taken outdoors it registers 62° .

(a) Calculate the outdoor temperature.

(b) Assuming the thermometer remains outdoors (where the temperature is constant) when will it register 51° ?

6. A veterinarian about to engage in surgery on a dog estimates that it will take him 45 minutes to complete the procedure. If 20 mg. of sodium pentobarbitol per kilogram of body weight is needed to barely maintain anesthesia, if the half-life of the anesthetic is five hours in dogs, and if the dog weighs 20 kilograms, how much anesthetic should the doctor administer initially to maintain anesthesia over the estimated duration of the operation? At the end of the 45 minute period the doctor realizes that the dog is beginning to emerge from anesthesia and that the surgery will take a half-hour longer to complete. What dosage of pentobarbitol should be administered at that point?

7. Consider a cigarette being used as a filter for some component (say nicotine) in the smoke. For a linear filter, the amount picked up by the filter over a small length is proportional to the concentration C and to the length Δx .



Show that the filtration equation is the familiar decay equation

$$\frac{dC}{dx} = -kC$$

and that

$$C = C_0 e^{-kx}$$

[Note: We are assuming here that the cigarette has not burnt down appreciably. For this case, we have a moving boundary problem which leads to a partial differential equation.]

8. Electrical circuits offer several examples of decay processes. The fundamental electrical quantities are charge q, current

(1) $$I = \frac{dq}{dt},$$

and voltage or electromotive force V. The electrical circuit components, resistors, condensers, and coils, have electrical properties measured by certain constants, resistance R, capacity C, and inductance L, respectively. The capacity of a condenser is defined as the ratio of the charge on the condenser to the voltage required to produce it:

(2) $$C = \frac{q}{V}.$$

The resistance of a resistor is defined as the ratio of the imposed voltage to the current it produces:

(3) $$R = \frac{V}{I}.$$

In addition, we need to know Kirchhoff's rule, that the sum of the voltages across the elements of a circuit is zero.

(a) If a condenser at voltage V discharges across a resistor, by Kirchhoff's rule the voltage across the resistor is $-V$. Employ (1), (2), and (3) to obtain a differential equation for V and solve for V as a function of time subject to the condition $V = V_0$ at $t = 0$.

(b) A coil resists change in current in the same way that a resistor opposes the passage of charge. If the voltage V is applied to a coil the current changes at the rate

(4) $$\frac{dI}{dt} = \frac{V}{L}.$$

If we impose an external voltage E (say by means of a battery) upon a circuit consisting of a resistor and a coil, then by Kirchhoff's rule the sum of the voltages is zero; namely, from (2) and (3)

$$E - IR - L\frac{dI}{dt} = 0$$

whence,

(5) $$\frac{dI}{dt} = \frac{E}{L} - \frac{R}{L} I.$$

Solve (5) subject to the initial condition $I = 0$ at $t = 0$. (Hint: Express the solution as the sum of two terms, $I = \frac{E}{R} + J(t)$, where $\frac{E}{R}$ is the steady current which would be set up if the coil were not present, and $J(t)$ is a "transient" term which represents the effect of the coil.)

(c) If there is no external voltage and the current has the value $I = I_0$ at $t = 0$, verify that (5) is the equation of a simple decay process and obtain the solution of (5) under these conditions.

(d) Use Kirchhoff's rule to derive the equation

$$\frac{dE}{dt} - \frac{I}{C} - R\,\frac{dI}{dt} = 0$$

where a source of electromotive force, E, is connected in a series circuit to a resistor, R, and a condenser, C.

Determine I in terms of t if the electromagnetic force E is constant and the current is 0 at $t = 0$.

9. Consider the differential equation

$$\frac{dx}{dt} = a - bx$$

subject to the initial condition

$$x = x_0 \quad \text{at} \quad t = 0 .$$

(a) Show that the solution is given by

$$x = \frac{a}{b}(1 - e^{-bt}) + x_0 e^{-bt} .$$

(b) Show that if $x_0 = 0$ then the result has the same form as that of Number 8(b).

10. (a) Assume that the rate of inversion of raw sugar is proportional to the amount of raw sugar remaining. If after 4 hours, 500 pounds of raw sugar have been reduced to 200 pounds, how much raw sugar will remain at the end of 12 hours?

(b) Consider a chemical reaction in which the velocity of the reaction (or rate of change in the amount of the substance consumed) is proportional to the quantity of the unconsumed substance at that instant. Let x_0 be the quantity of the substance at $t = 0$ and x the quantity that has been converted by time t. Show that

$$x = x_0\,(1 - e^{-kt})$$

where k may be determined from the amount of substance x_1 consumed by time t_1:

$$k = \frac{1}{t_1} \log \frac{A}{A - x_1} .$$

11. One dollar silver certificates (payable in silver on demand) are being retired from circulation and replaced by Federal Reserve notes (pure paper). Let N be the number of silver certificates and M the number of Federal Reserve notes in circulation at time t and suppose that the two varieties are uniformly mixed together. Suppose that the number of dollar bills passing through the central banks each day is a constant k, and that all silver certificates among these are replaced by Federal Reserve notes. Suppose that the process is initiated at time $t = 0$ when $N = N_0$ and $M = 0$, and that the total number of dollar bills in circulation is held constant. Determine the way N and M depend on time (measured in banking days) and find the number of days it takes to replace half the silver certificates.

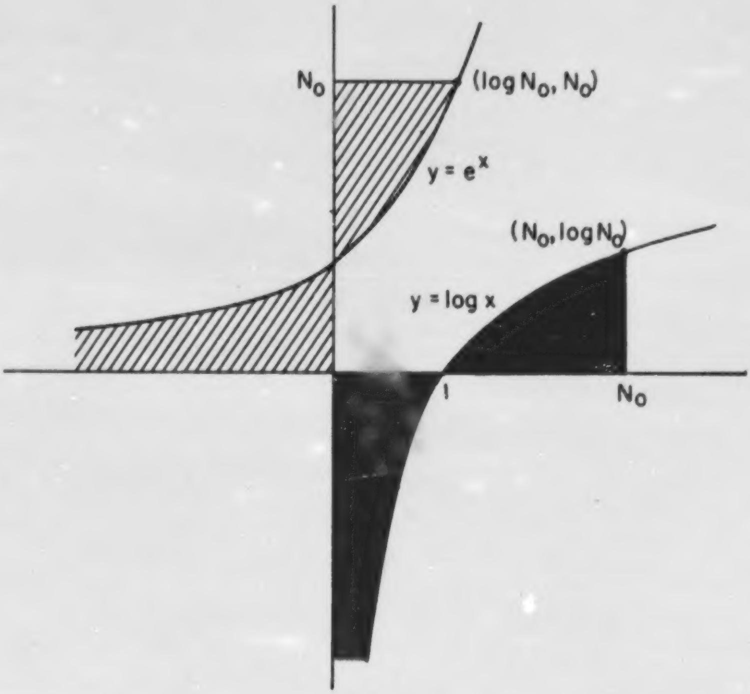
12. The principle of dating organic matter by radioactive carbon content is based on the observation that the ratio of the concentration of radioactive carbon C^{14} to that of ordinary carbon C^{12} in atmospheric carbon dioxide is maintained at a constant level because of continual cosmic ray bombardment. An organism throughout its life takes up carbon in the same proportions, but after death the relative amount of C^{14} decreases because of radioactive decay without replenishment. Let τ be the half-life of C^{14}. Show how to date an ancient timber if the ratio of C^{14} to C^{12} in the specimen is known.

13. The continuous model for radioactive decay used here replaces the picture of discrete atoms disintegrating at random times. The utility of such a model depends upon the involvement of a large number of atoms in the process. Reflect realistically about the model. Is it true that a mass of radioactive substance can never completely decay? If not, offer a reasonable estimate of the time of complete disappearance of the substance.

14. Combining Equations (3) and (4) of Section 9-3(i) we obtain for the mean life-time in radioactive decay

$$T = -\frac{1}{AN_0}\int_0^{N_0} \log N \, dN + \frac{1}{A}\log N_0 .$$

The integral $\int_0^{N_0} \log N \, dN$ is the sum of the signed areas of the shaded regions bounded partly by the graph of $y = \log x$ in the following figure.



It is geometrically clear that the areas of the shaded regions are equal to the areas of the hatched regions bounded partly by the graph of $y = e^x$. From this, calculate the integral and hence obtain the mean life-time given in the text.

15. Bacteria and other cells reproduce by splitting in two. What is the average time between "birth" and cell division for a member of the bacterial colony of Exercises 9-2, Number 1?

16. Here is an example of decay which is not exponential. A spherical moth ball in a closet evaporates away at a rate proportional to its surface area. If half of it (in weight) evaporates away in 10 days, determine how many more days it takes so that one quarter of the original amount is left.

17. Consider the differential equation

(1) $$(D_t + a)u = k\,e^{-bt}$$

(compare Equation (5) of Section 9-3(iii)).

(a) Observe for any solution u of (1) that

(2) $$(D_t + b)(D_t + a)u = 0$$

and show for any solutions v, w of the equations

(3) $$(D_t + b)v = 0\ ,\ (D_t + a)w = 0$$

that

(4) $$u = v + w$$

is a solution of (2). (Hint: $(D_t + b)(D_t + a) = (D_t + a)(D_t + b)$.)

(b) Use the form (4) to obtain a solution of (1) satisfying the initial condition

(5) $$u = u_0 \quad \text{at} \quad t = 0\ .$$

(c) Show that the solution of (1) satisfying the initial condition (5) is unique.

18. The differential Equation (1) of the preceding problem may also be solved as follows. Let v be a solution of the homogeneous equation

$$(D_t + a)v = 0$$

and determine w such that $u = v \cdot w$ is a solution of (1). Following this procedure, determine again the solution of (1) satisfying the initial condition (5) of the preceding problem.

19. (a) Employ the method of Number 18 to obtain the solution (9) of Equation (1) in Section 9-3(iv) subject to the initial condition $N(0) = N_0$; that is, seek a solution of the form $N(t) = U(t)W(t)$ where $U(t)$ is a solution of the homogeneous equation

$$\frac{dU}{dt} + AU = 0$$

(b) Verify that the solution is unique.

Exercises 9-5

1. A reaction in which one molecule each of the reagents A_1, A_2, A_3, ..., A_n combine to form one molecule of the product B is indicated by

$$A_1 + A_2 + \dots + A_n \xrightarrow{k} B$$

where k is the rate constant in the differential Equation (12) of Section 9-4(iv). We write (12) in the form

$$\frac{db}{dt} = ka_1a_2a_3 \dots a_n$$

where the lower case letters denote the concentrations of the corresponding reagents and the product. If more than one molecule of a given reagent enters into a reaction as for the familiar example

$$2H_2 + O_2 \longrightarrow 2H_2O \tag{1}$$

then the concentration of a given reagent enters the law of mass action in as many places as the number of its molecules which are involved in building one molecule of the product. Thus, for a reaction in the form of (1)

$$2A_1 + A_2 \longrightarrow B$$

the law of mass action takes the form

$$\frac{db}{dt} = k\, a_1^2\, a_2 . \tag{2}$$

(a) Write the law of mass action (2) in the form corresponding to (12) of Section 9-4(iv). What are the rates $\frac{da_1}{dt}$ and $\frac{da_2}{dt}$? (Hint: Use the fact that the amount of each element is unaltered in a chemical reaction).

(b) Similarly, for a reaction in which ν_i molecules of the reagent A_i are combined to form one molecule of the product B, we write

$$\nu_1A_1 + \nu_2A_2 + \dots + \nu_nA_n \xrightarrow{k} B$$

and obtain the law of mass action in the form

$$\frac{db}{dt} = ka_1^{\nu_1}\, a_2^{\nu_2} \dots a_n^{\nu_n} . \tag{3}$$

Write the law of mass action (3) in the form corresponding to (12). Determine $\frac{da_i}{dt}$ in terms of a_i.

2. When a reaction product is obtained as the result of a chain of reactions the law of mass action cannot be applied directly to obtain the rate of production. We must take account of the intermediate reactions. The simplest example is given by a reversible unimolecular reaction in which a molecule of A may be converted to a molecule of B with one probability, but a molecule of B may revert to A with another probability. This reaction is indicated by

$$A \underset{k_2}{\overset{k_1}{\rightleftharpoons}} B$$

and the reaction is governed by the equation

$$\frac{db}{dt} = k_1 a - k_2 b \tag{1}$$

where k_1 is the rate constant for conversion of A to B and k_2 is the rate constant for conversion of B to A . Let C_0 denote the initial concentration of B and C_1 the initial concentration of A , and write (1) in the form of (12) of Section 9-4(iv). Describe the course of the reaction. What state is approached as t approaches infinity?

3. A catalyst is a compound which enters into a reaction, but which is not consumed in the process. The effect of employing a catalyst is to enhance the reaction. Letting E denote the catalyst, the form of a simple catalyzed reaction is

$$E + A \xrightarrow{k} E + B .$$

Since the catalyst is not consumed in the reaction, its concentration $\mathcal{E}$ remains constant and therefore we expect

$$\frac{db}{dt} = k\,\mathcal{E}\,a = k\,\mathcal{E}\,(C - b) \tag{1}$$

where C is the initial concentration of A .

Life is never so simple; the action of biological catalysts or enzymes does not follow the elementary law (1). It was conjectured by Michaelis and Menten* that a reagent or substrate A enters into a loosely bound intermediate combination with the enzyme E . The intermediate combination may dissociate into E and A or into E and the reaction product

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*See Bartholomay, A.F., "Physicomathematical foundations of reaction rate theory" in Physicomathematical Aspects of Biology, Academic Press, N.Y., 1962.

B . Thus an enzymatically catalyzed reaction is assumed to take the form

$$E + A \underset{k_2}{\overset{k_1}{\rightleftharpoons}} I \xrightarrow{k_3} E + B \text{ .}$$

Verify that a reaction of this form satisfies the algebraic relations

$$\mathcal{E} + I = \mathcal{E}_0 \text{ , } a + i + b = C \text{ ,} \tag{2}$$

where a , $\mathcal{E}$, i , b denote the concentrations of substrate, enzyme, intermediate, and product, respectively, and at $t = 0$ the initial concentrations are given by

$$\begin{aligned} &a(0) = C \text{ , } \mathcal{E}(0) = \mathcal{E}_0 \text{ ,} \\ &i(0) = 0 \text{ , } b(0) = 0 \text{ .} \end{aligned} \tag{3}$$

Show that the reaction is governed by the differential equations

$$\begin{aligned} \frac{da}{dt} &= - k_1 \mathcal{E} a + k_2 i \\ \frac{d\mathcal{E}}{dt} &= - k_1 \mathcal{E} a + (k_2 + k_3) i \\ \frac{di}{dt} &= k_1 \mathcal{E} a - (k_2 + k_3) i \\ \frac{db}{dt} &= k_3 i \text{ .} \end{aligned} \tag{4}$$

The system (3), (4) has not been solved explicitly. In order to obtain an explicit practical solution Briggs and Haldane assumed that the initial amount of substrate C is very large compared to the amount $\mathcal{E}_0$ of available enzyme, and that the concentration i reaches near equilibrium before an appreciable amount of the substrate is converted. The equilibrium assumption requires that, to a good approximation,

$$\frac{di}{dt} = k_1 \mathcal{E} a - (k_2 + k_3) i = 0 \text{ .} \tag{5}$$

Ignoring the very rapid initial phase of the reaction in which the equilibrium (5) is set up and the terminal phase in which the concentration of substrate is not large compared to $\mathcal{E}_0$, we replace the differential system (4) by

$$(6) \qquad \frac{da}{dt} = -k_1 \mathcal{E} a + k_2 i \qquad \frac{db}{dt} = k_3 i \,,$$

the algebraic relations (2) by

$$(7) \qquad \mathcal{E} + i = \mathcal{E}_0 \,, \quad a + i + b = C \,, \quad \mathcal{E} a = k_M i$$

where k_M is the Michaelis constant $k_M = \frac{(k_2 + k_3)}{k_1}$ and the last equation comes from (5). Compatibly with the first of the assumptions above we replace the initial conditions (3) by

$$(8) \qquad a(0) = C_1 \,, \ \mathcal{E}(0) = C_2 \,, \qquad i(0) = C_3 \,, \ b(0) = 0 \,.$$

Here the constants C_1, C_2, C_3 must be compatible. Thus, the constant C_3 corresponds to the relatively small amount of A which is consumed in reaching the equilibrium (5).

The argument we have given in obtaining the modified system (6), (7), (8) is not mathematically complete. To complete the story it is necessary to show how well the solution of the modified system approximates that of the original system. This is a more difficult question which we shall not attempt to answer here. The main point is that the modified system is easy to solve explicitly while the original system, which incorporated all the information at our disposal, is not. In practice the modified system is therefore actually more useful and under the stipulated assumption that a is much greater than $\mathcal{E}_0$ it has proved to be adequate.

Solve the system (6), (8). (Hint: Use the first and third equations in (7) to eliminate and then obtain the differential equation for a.) Why is it unnecessary to obtain the differential equation for b in order to solve for b ?

4. A Soviet statistical yearbook reports that the birthrate was 21.2 per 1000 in 1963 and the deathrate 7.2 per 1000 . As of Jan. 1, 1965 the population was officially estimated at 229 million. The official Soviet population projection predicts a population of 250 million in 1970, 263 million in 1975, and 280 million in 1980. Assuming that the projection is based on a constant deathrate, show that the published report predicts an immediate upswing in the country's birthrate.

5. In this problem it may seem at first that we do not have enough data for its solution, but the situation can be handled in terms of differential equations.

   It began snowing sometime before noon. A snow plow set out to clear a road at noon. It traveled the first mile in one hour and the second mile in two hours. What time did it start snowing? (Assume the snow falls at a constant rate (ft/hr) and that the snow plow removes snow at a constant rate (ft^3/hr), and neglect the compressibility of the snow.)

Chapter 10

INTEGRATION

10-1. Introduction.

It is no accident that problems in the applications, like those of Chapter 9, tend to be posed in the form of differential equations. Differential equations are relations between unknown functions and their derivatives. A differential equation refers to local properties: it describes events in the neighborhood of a given point or a given instant of time. Local behavior is easy to observe and lends itself readily to intelligent surmise. For example, we might guess without direct observation that the rate of spread of an epidemic in a community is proportional to the number of active infectious cases and to the number of individuals who have not yet been infected.

Often our primary concern is not local, but global; it may not be the differential equation which interests us most, but some properties which depend on a general knowledge of the solution. Our interest in the rate of growth of a bacterial population may be academic; but not our interst in whether the total population of bacteria in a host will reach a dangerous size before the host organism can marshal its defenses. Thus, the problems which concern us most are likely to be problems of integration, of obtaining the solutions of differential equations, or at least obtaining some specific information about these solutions.

This chapter is devoted primarily to the problem of formal integration for the differential equation

$$DF = f \tag{1}$$

where f is a known function. In principle, this problem is solved by the Fundamental Theorem. If f is continuous on an interval containing both a and x then the solution of (1) subject to the initial condition

$$f(a) = C \tag{2}$$

exists, is unique, and can be written in the form

$$F(x) = C + \int_a^x f(t)\,dt \ . \tag{3}$$

Formula (3) is no more than a symbolic representation of the solution. If possible, we should like to obtain a simple analytical formula for $F(x)$ in terms of objects with which we are familiar. Failing that, we should like to have a practically useful approximation to $F(x)$. We seek then for a way of representing the integrand in a form which is recognizable as the derivative of some known function. It is not always possible to do so, but then we may seek a representation which is more amenable to approximation. In this chapter, we shall explore some of the methods for transforming integrals into more convenient forms; these methods are the so-called "techniques of integration."

There are two basic analytical techniques, the methods of substitution and of integration by parts. In applying these methods we either transform the integral so that the integrand is recognizable as the derivative of a familiar function, or transform it into another integral which is more manageable. We shall see that many integration problems may be reduced to the integration of rational functions. For these there exists a special algebraic technique of decomposition into partial fractions which permits an immediate integration. There is a wealth of special techniques but we treat only the most important.

The functions we have dealt with in this text are called elementary functions. What is or is not an elementary function is a matter of somewhat arbitrary definition: it is a collection of functions which is useful, and with which we are familiar. For our present purposes, the elementary functions consist of the powers, the circular functions, the logarithmic and exponential functions, and all functions obtained from these by rational combination, inversion, and composition. We have demonstrated in Chapters 4 and 8 that the derivative of an elementary function is again an elementary function. An indefinite integral of an elementary function is not necessarily an elementary function. That fact is not proved here,[*] but it is emphasized so that you will not be misled by our success in representing many integrals in terms of elementary functions.

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[*] Sec. G.H. Hardy, The Integration of Functions of a Single Variable, University Press, Cambridge, 1916.

The purpose of this chapter is not to make you a master in the art of integration in terms of elementary functions. It is enough for you to make use of one of the tables of integrals which provide solutions of such integration problems catalogued in some more-or-less systematic fashion (but be sure to check against possible errors and misprints). Nonetheless, a knowledge of the techniques and reasonable proficiency in their use are desirable to facilitate theoretical and numerical analysis. Furthermore, the tables contain only certain standard forms and even if you wish only to use the tables efficiently it is necessary to learn how to transform an integral of concern to you into one of the standard forms.

In this chapter our objective is to give a method for expressing integrals from certain broad classes in terms of a few basic elementary integrals. For easy reference, we list here the principle integration formulas at our disposal from Chapters 4 and 8.

Table 10-1a

$f(x) = F'(x)$	$F(x) = C + \int_{x_0}^{x} f(t)dt$
(1) a (a constant)	ax
(2) x^r (r real, $r \neq -1$)	$\frac{x^{r+1}}{r+1}$
(3) $\frac{1}{x}$, $x > 0$	$\log x$
(4) e^x	e^x
(5) $\sin x$	$-\cos x$
(6) $\cos x$	$\sin x$
(7) $\frac{1}{\cos^2 x} = 1 + \tan^2 x$	$\tan x$
(8) $\frac{1}{\sqrt{1 - x^2}}$, $\lvert x \rvert < 1$	$\arcsin x$
(9) $\frac{1}{1 + x^2}$	$\arctan x$

We have not stressed the hyperbolic functions and their inverses, since they can be expressed in terms of exponentials and logarithms. However, these functions appear commonly in tables and in mathematical literature. We introduce the inverse hyperbolic functions

$$\begin{aligned} &\text{arg sin h} : \sinh x \longrightarrow x \ , \\ &\text{arg cosh} : \cosh x \longrightarrow x \ , \ x \geq 0 \ , \\ &\text{arg tanh} : \tanh x \longrightarrow x \ , \ |x| < 1 \ , \end{aligned}$$

etc., where arg in each case denotes the argument of the corresponding hyperbolic function. The integrals of some important algebraic functions can be expressed in terms of inverse hyperbolic functions. These are listed below and left for you to verify.

Table 10-1b

$f(x) = F'(x)$	$F(x) = C + \int_{x_0}^{x} f(t)dt$
(10) $\frac{1}{1 - x^2}$	$\frac{1}{2} \log\left\|\frac{1 + x}{1 - x}\right\|$ $= \text{arg tanh } x$, if $\|x\| < 1$ $= \text{arg coth } x$, if $\|x\| > 1$
(11) $\frac{1}{\sqrt{x^2 + 1}}$	$\log(x + \sqrt{x^2 + 1}) = \text{arg sinh } x$
(12) $\frac{1}{\sqrt{x^2 + 1}}$	$\log \|x + \sqrt{x^2 - 1}\|$ $= \text{arg cosh } x$, if $x > 1$ $= -\text{arg cosh } \|x\|$, if $x < -1$

It is not essential to memorize the formulas of Table 10-1b, they can be obtained easily by the substitution of a hyperbolic function for x so described in Section 10-3.

Exercises 10-1

1. For each of the following sketch the graph of f defined by the given integral.

   (a) $f(x) = \int_0^x t^3 e^{-t} dt \ , \ x \geq 0$

   (b) $f(x) = \int_0^x (2 - t)e^{-t} dt \ , \ x \geq 0$

   (c) $f(x) = \int_2^x (4 - t^2) dt$

   (d) $f(x) = \int_1^x t \log t \ dt \ , \ x > 0$

2. Improve the sketches of Number 1 by employing the following information for corresponding parts (a) - (d):

   (a) $f(3) = 6 - \frac{78}{e^3} \ , \ \lim\limits_{x \sim \infty} f(x) = 6$.

   (b) $f(2) = 1 + \frac{1}{e^2} \ , \ f(3) = 1 + \frac{2}{e^3} \ , \ \lim\limits_{x \sim \infty} f(x) = 1$.

   (c) supply any needed extra information by evaluating the integral.

   (d) $\lim\limits_{x \sim 0^+} f(x) = \frac{1}{4} \ , \ f(\frac{1}{e}) = \frac{1}{4}(1 - \frac{3}{e^2})$.

3. Sketch the graph of, $f : x \longrightarrow \int_0^x \sin \pi t \ dt$.

4. Let $f(x) = \int_0^x \cos^4 t \ dt$. Given that $f(\frac{\pi}{2}) = \frac{3\pi}{16}$ and $f(\pi) = \frac{3\pi}{8}$, sketch the graph of f . (Hint: show first that for all x , $f(x + \pi) = a + f(x)$.)

5. Determine constants x_0 and C for the column on the right in Table 10-1a for each function f , so that $F(x) = C + \int_{x_0}^x f(t) dt$.

6. Verify the integration formulas of Table 10-1b and determine appropriate constants x_0 and C . (Compare the results of Exercises 8-7, No. 9.)

10-2. The Substitution Rule.

The substitution rule is a rule for changing the parameter of integration. It is the integration formula corresponding to the chain rule of differentiation (Section 4-6).

THEOREM 10-2. (Substitution Rule). Let f be a continuous function and let F be an integral of f. Let g be a continuously differentiable function whose range lies in the domain of f, and let α and β be numbers in the domain of g. Then

(1) $$\int_\alpha^\beta f\big(g(t)\big)g'(t)dt = \int_a^b f(x)dx ,$$

where

(2) $$a = g(\alpha) \quad \text{and} \quad b = g(\beta) .$$

Proof. Set $H(t) = F\big(g(t)\big)$. Since F is an integral of f we have by the chain rule

$$H'(t) = F'\big(g(t)\big)g'(t) = f\big(g(t)\big)g'(t) .$$

It follows from the Fundamental Theorem that

$$H(\beta) - H(\alpha) = \int_\alpha^\beta f\big(g(t)\big)g'(t)dt .$$

Now, we observe also that

$$H(\beta) - H(\alpha) = F\big(g(\beta)\big) - F\big(g(\alpha)\big) = F(b) - F(a) = \int_a^b f(x)dx ,$$

from which the desired result follows.

The Leibnizian notation is particularly apt for the substitution rule, as it is for the chain rule. If we put

(3) $$x = g(t)$$

we obtain Equation (1) in the Leibnizian form

(4) $$\int_a^b f(x)dx = \int_\alpha^\beta f(x)\,\frac{dx}{dt}\,dt .$$

Here, to replace the parameter of integration x by the parameter t, we substitute $g(t)$ for x; for the values of x at the ends of integration we substitute the corresponding values of t, for the "differential" dx we substitute $\frac{dx}{dt}\,dt$. We use the symbolic relation

$$dx = \frac{dx}{dt}\,dt$$

to remember this substitution, but attach no meaning to this equation except as a formal rule of substitution.

Finally, from (4) we obtain the rule for the indefinite integrals,

$$\int f(x)dx = \int f(x)\,\frac{dx}{dt}\,dt\ , \qquad (x = g(t)) . \tag{5}$$

Example 10-2a. Consider the problem of integrating $\frac{1}{x}$ between negative limits, say, a and b , where $a < 0$, $b < 0$. For $x = -t$ we have by (4)

$$\begin{aligned}\int_a^b \frac{1}{x}\,dx = \int_{-a}^{-b} \frac{1}{t}\,dt &= \log\,(-b) - \log\,(-a) \\ &= \log\,|b| - \log\,|a| .\end{aligned}$$

For the indefinite integral of $\frac{1}{x}$ we therefore obtain,

$$\int \frac{1}{x}\,dx = \log\,|x| + C ,$$

which generalizes Formula (3) of Table 10-1a. (The formula for the indefinite integral of $\frac{1}{x}$ can be meaningfully applied to calculate the value of the definite integral only if a and b have the same sign.)

The substitution rule for indefinite integrals is employed in two different ways. We illustrate this with examples. The first application is direct. Suppose that we recognize the integral in the form $\int f\big(g(x)\big)g'(x)dx$ where f is one of the functions in Tables 10-1a, b whose antiderivative is F. The substitution rule tells us that this integral is equal to $\int f(t)dt = F(t) + C$ with $t = g(x)$, that is $F\big(g(x)\big) + C$.

Example 10-2b. Consider the problem of integrating $\frac{2x}{1 + x^2}$. We note that $2x = D(1 + x^2)$. For $t = g(x) = 1 + x^2$, $f(t) = \frac{1}{t}$, we get

$$\int \frac{2x}{1 + x^2}\,dx = \int \frac{g'(x)}{g(x)}\,dx = \int \frac{1}{t}\,dt = \log\,|t| + C = F(t) + C = \log(1 + x^2) + C .$$

It is important to recognize here that we seek a function H for which $H'(x) = f\big(g(x)\big)g'(x)$. The problem is not completely solved until the answer is expressed in terms of the original parameter x ; that is, the answer is not $F(t) + C$ but $H(x) + C = F\big(g(x)\big) + C$. (In Example 10-2b, $H(x) = \log(1 + x^2)$.)

Example 10-2c. Next consider the problem of integrating the function $x\sqrt{1 - x^2}$. Observing that x is proportional to the derivative of $1 - x^2$, we set $t = 1 - x^2$ and obtain

$$\int x \sqrt{1 - x^2}\, dx = \int \sqrt{t}\, (- \frac{1}{2} \frac{dt}{dx}) dx = - \frac{1}{2} \int \sqrt{t}\, dt$$
$$= - \frac{1}{3} t^{3/2} + C = - \frac{1}{3} (1 - x^2)^{3/2} + C \ .$$

In the preceding example one of the factors depended upon the expression $1 - x^2$ and the other factor, apart from a constant multiplier, was the derivative of that expression, i.e., the integrand was given in the form $k\, f\big(g(t)\big)g'(t)$, k constant. In that case, if F is an integral of f , we may immediately recognize the integral as $k\, F\big(g(t)\big)$. (Caution: always check your integrations by differentiating; it is pathetically easy to forget a constant factor.)

Example 10-2d. Let us integrate $\tan \theta$. Observe that $\tan \theta = \frac{\sin \theta}{\cos \theta}$ so that the numerator is the negative of the derivative of denominator. Consequently, applying the preceding remark and employing the result of Example 10-2a, we get

$$\int \tan \theta \, d\theta = -\log |\cos \theta| + C \ .$$

As the preceding examples show, integration is based to a large extent on the art of observation. Like any other algorithmic skill it requires the recognition of the structure of a formula beneath its details.

Example 10-2e. In examining the integral

$$I = \int \frac{x^2}{\sqrt{1 - x^6}}\, dx$$

we recognize x^3 as the "structural unit." Setting $u = x^3$ and $du = 3x^2 dx$, we obtain

$$I = \frac{1}{3} \int \frac{du}{\sqrt{1 - u^2}} = \frac{1}{3} \arcsin u + C = \frac{1}{3} \arcsin x^3 + C \ .$$

The second way of applying the substitution rule is in the reverse direction to the first and is an exploratory device. Suppose we want to find $\int f(x)dx$ but f does not look like a derivative of a function that we know. Using an educated guess, we pick some differentiable function u (which we recognize as a structural element in the expression for $f(x)$) with the hope that upon the substitution $x = u(t)$, $f\big(u(t)\big)u'(t)$ is the derivative of some known function H . If u has an inverse ψ , our aim is then achieved since the substitution rule gives

$$\int f(x)dx = \int f\big(u(t)\big)u'(t)dt = H(t) + C = H\big(\psi(x)\big) + C .$$

Usually, a further manipulation is needed in order to recognize the integrand in proper form, as in the following example.

Example 10-2f. In Exercises 9-3, Number 14 we showed how to integrate $\log x$ with the aid of a geometrical argument and the known integral of the inverse function. We now attack the problem analytically using the substitution $x = e^t$, $dx = e^t dt$. We obtain

$$\int \log x\, dx = \int t\, e^t\, dt .$$

We may not immediately recognize te^t as a familiar derivative, but we can experiment. The derivative of te^t is not quite te^t itself, but

$$D(t\, e^t) = t\, e^t + e^t = t\, e^t + D\, e^t .$$

Consequently,

$$t\, e^t = D(t\, e^t - e^t) .$$

With this observation our problem is solved:

$$\int t\, e^t\, dt = e^t(t - 1) + C = x(\log x - 1) + C .$$

Later, using the method of integration by parts we shall be able to treat such problems systematically.

In using the exploratory method we usually do not substitute directly for x , but pick an expression in the integrand, say $u(x)$, which appears particularly troublesome and set $u(x) = t$, assuming that u has an inverse ψ . This amounts effectively to the substitution $x = \psi(t)$. In Example 10-2f the troublesome term was $\log x$ and we set $\log x = t$.

Example 10-2g. Consider the integral

$$I = \int \frac{1}{\sqrt{x} + \sqrt[3]{x}}\, dx$$

In order to eliminate the two radicals in the denominator we set $\sqrt[6]{x} = t$. Then $\sqrt{x} = t^3$ and $\sqrt[3]{x} = t^2$. Since $x = t^6$, $dx = 6t^5\, dt$ and we get

$$I = \int \frac{1}{t^2 + t^3} (6\, t^5)\, dt = 6 \int \frac{t^3}{1 + t}\, dt \, .$$

Still we do not recognize a derivative of a known function. The troublesome term is now $(t + 1)$ in the denominator, so we make a second substitution $t + 1 = s$ from which $t = s - 1$, $dt = ds$. Thus

$$\begin{aligned} I &= 6 \int \frac{t^3}{1 + t}\, dt = 6 \int \frac{(s - 1)^3}{s}\, ds = 6 \int (s^2 - 3s + 3 - \tfrac{1}{s})\, ds \\ &= 2s^3 - 9s^2 + 18s - 6 \log |s| + C_1 \\ &= 2t^3 - 3t^2 + 6t - 6 \log |1 + t| + 11 + C_1 \\ &= 2x^{1/2} - 3x^{1/3} + 6x^{1/6} - 6 \log(1 + x^{1/6}) + C_2 \, . \end{aligned}$$

Exercises 10-2

1. Integrate in terms of elementary functions when possible.

(a) $\dfrac{x^2}{x^3 + a^3}$

(b) $x^3 \sqrt[7]{1 - x^4}$

(c) $\dfrac{x - 1}{\sqrt{x^2 - 2x + a^2}}$

(d) $\dfrac{(a + b\sqrt{x})^{13}}{\sqrt{x}}$

(e) $\dfrac{x^{m-1}}{ax^m + b}$

(f) $\dfrac{1}{x + x^\alpha}$, $\alpha \neq 1$

(g) $\dfrac{x^2 + 1}{x - 1}$

(h) $\dfrac{(1 + x)^2}{1 + x^2}$

(i) $\dfrac{1}{\sqrt{1 - x^4}}$

(j) $\dfrac{x}{\sqrt{a^4 - x^4}}$

(k) $\dfrac{x}{\sqrt{a^4 + x^4}}$

(ℓ) $\dfrac{x}{x^4 + a^4}$

2. Integrate in terms of elementary functions.

(a) $\frac{\sin x}{(a + b \cos x)^n}$

(b) $3 \cos x \sin 2x$

(c) $\sin^\alpha x \cos x$

(d) $\sin^m x \cos^3 x$

(e) $\sin^2 2x$

(f) $\sec^5 ax \tan ax$

(g) $\frac{\cos \sqrt{2x}}{\sqrt{x}}$

(h) $\sqrt{\frac{\cos x}{\tan x}}$

(i) $\frac{1}{a + b \cos^2 x}$, $a \neq 0$

3. Integrate in terms of elementary functions.

(a) $2x^3 e^{x^4}$

(b) $e^x (a + b e^x)r$

(c) $e^x (a + b e^{-x})^3$

(d) $\frac{ae^x}{b + ce^x}$, $c \neq 0$

(e) $\frac{a}{b + ce^x}$, $c \neq 0$

(f) $\frac{\log x}{x}$

(g) $\frac{1}{x \log x}$

(h) $\frac{\log^2 x}{x}$

(i) $\sinh^m ax \cosh ax$

(j) $\cosh^3 ax$

(k) $\frac{1}{\sinh x + 2 \cosh x}$

(ℓ) $\frac{1}{a^2 \sinh^2 x + b^2 \cosh^2 x}$, $b \neq 0$

(m) $a^x e^{(a^x)}$

4. Integrate in terms of elementary functions.

(a) $\frac{x^2}{\sqrt[5]{x + a}}$

(b) $\frac{1}{\sqrt[3]{x} - \sqrt[4]{x}}$

(c) $\frac{1}{\sqrt[6]{x} + \sqrt[7]{2x}}$

(d) $\frac{\sin 2x}{1 + 4 \cos^2 x}$

(e) $x e^{x^2} e^{e^{x^2}}$

(f) $\frac{x + 1}{x(a + xe^x)}$

(g) $\frac{(\arctan x)^2}{1 + x^2}$

(h) $\frac{\log(\log x)^2}{x \log x}$

(i) $(x + 1)e^x \tan(xe^x)$

(j) $\frac{x e^{x^2} \cos [\log(e^{x^2} + 1)]}{e^{x^2} + 1}$

5. Use the idea of Example 10-2f to integrate $t^2 e^t$, $t^3 e^t$. Can you obtain and prove a formula for the indefinite integral of $t^n e^t$?

10-3. Substitutions of Circular Functions.

Although it is not always possible to integrate a given function in terms of elementary functions, there are important broad classes of explicitly integrable functions. All powers and hence, clearly, all polynomials are explicitly integrable. It is not so clear but it is true that all rational functions are explicitly integrable (see Section 10-6). It follows that all integrals which can be transformed by substitution into integrals of rational functions are explicitly integrable. In this section we shall show that an integral of any rational combination of x and $\sqrt{Q(x)}$, where

$$Q(x) = Ax^2 + Bx + C ,$$

can be transformed into an integral of a rational combination of circular functions, and further that an integral of a rational combination of circular functions can be transformed into an integral of a rational function.

We should consider the substitution of a circular function whenever an integrand is a combination of x and one of the expressions $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, $\sqrt{x^2 - a^2}$, $(a > 0)$ suggestive of the Pythagorean expression for one of the sides of a right triangle in terms of the other two.

Example 10-3a. Consider

$$I = \int_0^{a/2} \frac{dx}{\sqrt{a^2 - x^2}} .$$

We utilize the substitution

$$x = a \sin \theta , \sqrt{a^2 - x^2} = a \cos \theta \qquad \left(-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right) ,$$

$$dx = a \cos \theta \, d\theta .$$

(See Figure 10-3a.) Observing that for $x = \frac{a}{2}$, $\theta = \frac{\pi}{6}$, we obtain by the substitution rule,

$$I = \int_0^{\pi/6} \frac{a \cos \theta}{a \cos \theta} d\theta = \int_0^{\pi/6} d\theta = \frac{\pi}{6} .$$

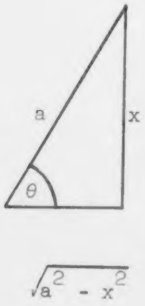

Figure 10-3a

Example 10-3b. For the integral

$$I = \int \frac{dx}{(x^2 + a^2)^{3/2}}$$

we employ the substitution (see Figure 10-3b)

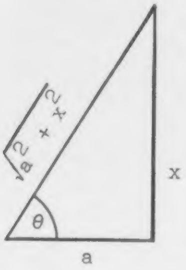

Figure 10-3b

$$x = a \tan \theta \qquad (-\frac{\pi}{2} < \theta < \frac{\pi}{2}) ,$$

$$dx = \frac{a}{\cos^2 \theta} d\theta$$

$$\sqrt{a^2 + x^2} = \frac{a}{\cos \theta} .$$

Thus we obtain

$$I = \int \frac{\cos^3 \theta}{a^3} \frac{a}{\cos^2 \theta} d\theta = \frac{1}{a^2} \int \cos \theta \, d\theta$$

$$= \frac{\sin \theta}{a^2} + C = \frac{x}{a^2 \sqrt{a^2 + x^2}} + C .$$

Example 10-3c. The integration

$$I = \int \frac{1}{x^2 \sqrt{x^2 - a^2}} dx$$

is performed with the aid of the substitution (see Figure 10-3c)[*]

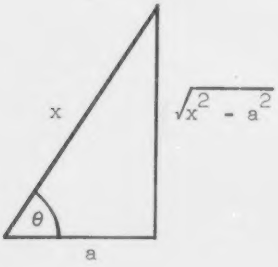

Figure 10-3c

$$x = \frac{a}{\cos \theta} ,$$

$$dx = \frac{a \sin \theta}{\cos^2 \theta} d\theta ,$$

$$\sqrt{x^2 - a^2} = a \tan \theta .$$

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[*] Here take $0 < \theta < \frac{\pi}{2}$ for $x > 0$, and $\frac{\pi}{2} < \theta < \pi$ for $x < 0$.

We have

$$I = \int\left(\frac{\cos^2\theta}{a^2}\right)\left(\frac{1}{a\tan\theta}\right)\left(\frac{a\sin\theta}{\cos^2\theta}\right)d\theta$$

$$= \frac{1}{a^2}\int\cos\theta\,d\theta = \frac{\sin\theta}{a^2} + C = \frac{\sqrt{x^2-a^2}}{a^2x} + C\ .$$

It is frequently simpler to use hyperbolic functions rather than triogonometric functions for integrals of the types considered above. If the substitution of a circular function leads to complications, try a hyperbolic substitution instead.

Example 10-3d. Consider the integral

$$I = \int\frac{1}{\sqrt{x^2-a^2}}\,dx\ .$$

Using the substitution of Example 10-3c we obtain

$$I = \int\frac{1}{a\tan\theta}\left(\frac{a\sin\theta}{\cos^2\theta}\right)d\theta = \int\frac{1}{\cos\theta}\,d\theta\ .$$

To complete the job algebraic trickery is needed (the objective of the manipulations will be clearer after Section 10-6 on decompositions into partial fractions). We have

$$\frac{1}{\cos\theta} = \frac{\cos\theta}{\cos^2\theta} = \frac{\cos\theta}{1-\sin^2\theta} = \frac{\cos\theta}{2}\left[\frac{1}{1-\sin\theta}+\frac{1}{1+\sin\theta}\right]\ .$$

With this much as a hint we leave the integration as an exercise.

On the other hand, if we had used the hyperbolic substitution*

$$x = a\cosh t\ ,\ \sqrt{x^2-a^2} = a\sinh t\ ,$$

$$dx = a\sinh t\,dt\ ,$$

we would have found immediately, by Formula (12) of Table 10-1b,

$$I = \int dt = t + C = \log\left|\frac{x+\sqrt{x^2-a^2}}{a}\right| + C = \log\left|x+\sqrt{x^2-a^2}\right| + D\ .$$

(See Exercises 8-7, No. 9.)

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*Valid for $x \geq |a|$. For $x \leq -|a|$ use $x = -a\cosh t$.

THEOREM 10-3a. An integral of any rational combination of x and $\sqrt{Q(x)}$ where

(1) $$Q(x) = Ax^2 + Bx + C \,, \qquad (A \neq 0)$$

can be transformed by a substitution $x = f(\theta)$, where f is a circular function, into an integral of a rational combination of $\sin\theta$ and $\cos\theta$.

Proof. We are concerned with integrals of the form

(2) $$I = \int \phi\left(x, \sqrt{Q(x)}\right) dx$$

where ϕ is a rational expression and $Q(x)$ is given by (1). For the proof we first make a preliminary linear transformation to replace $Q(x)$ by one of the standard forms of Examples 10-3a, b, c.

We "complete the square" to obtain

(3) $$Q(x) = A\left[\left(x + \frac{B}{2A}\right)^2 + \left(\frac{C}{A} - \frac{B^2}{4A^2}\right)\right] .$$

We set $a = \sqrt{\left|\frac{C}{A} - \frac{B^2}{4A^2}\right|}$, $b = \frac{B}{2A}$, $c = \sqrt{|A|}$, and $x = u - b$ in (3), and separate the problem into three cases.

Case (i).

If $A < 0$ and $\frac{C}{A} - \frac{B^2}{4A^2} < 0$ we have

$$\sqrt{Q(x)} = c\sqrt{a^2 - u^2} .$$

Since $dx = du$, the substitution $x = u - b$ yields

(4) $$I = \int \phi(u - b \,,\, c\sqrt{a^2 - u^2})du .$$

Now, employing the substitution $u = a\sin\theta$ of Example 10-3a, we transform the integral into the form

(5) $$I = a\int \phi(a\sin\theta - b \,,\, c\,a\cos\theta)\cos\theta \, d\theta \,, \quad \theta = \arcsin\frac{x + b}{a} .$$

Since ϕ involves only rational operations, we have established the theorem in this case.

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Case (ii).

If $A > 0$ and $\frac{C}{A} - \frac{B^2}{4A^2} < 0$, the substitution

$$x + b = u = a \tan \theta ,$$

as in Example 10-3b, confirms the theorem for this case.

Case (iii).

If $A > 0$ and $\frac{C}{A} - \frac{B^2}{4A^2} > 0$, the substitution

$$x + b = u = \frac{a}{\cos \theta} ,$$

as in Examples 10-3c, yields the desired result.

The integral (2) can be also transformed into an integral of a rational combination of $\sinh t$ and $\cosh t$ by an appropriate transformation $x = f(t)$ where f is a hyperbolic function. The proof is left as an exercise.

THEOREM 10-3b. An integral of a rational combination of $\sin x$ and $\cos x$ can be transformed into an integral of a rational function by a suitable substitution.

Proof. We consider integrals of the form

$$\int \psi(\sin x , \cos x)dx \tag{8}$$

where ψ is a rational expression. We observe that $\sin x$ and $\cos x$ are rational expressions in $t = \tan \frac{x}{2}$; namely,

$$\sin x = \frac{2t}{1 + t^2} , \cos x = \frac{1 - t^2}{1 + t^2} . \tag{9}$$

Furthermore,

$$dx = d(2 \arctan t) = \frac{2}{1 + t^2} dt . \tag{10}$$

Consequently we may transform the integral (8) into the integral of a rational function by employing the substitution

$$x = 2 \arctan t ; \tag{11}$$

thus, entering (9) and (10) in (8) we obtain the integral in the form

$$\int \psi\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)\frac{2}{1+t^2}\,dt \ . \tag{12}$$

Theorems 10-3a and 10-3b do not necessarily point the way to the simplest method of integration for a function of one of the types considered here; they simply indicate a line of approach which is sure to work but may lead to enormous complication. Often some special device leads to the solution far more simply and directly.

Exercises 10-3

1. Integrate the following functions, the numbers a and b being positive.

(a) $\dfrac{\sqrt{a^2 - x^2}}{x^2}$

(b) $\dfrac{\sqrt{1 + x^2}}{x^4}$

(c) $x^2\sqrt{a^2 - x^2}$

(d) $\dfrac{1}{x^2\sqrt{x^2 - a^2}}$

(e) $\dfrac{x}{(x^2 + a^2)\sqrt{x^2 - b^2}}$

(f) $\dfrac{1}{(x^2 + a^2)\sqrt{a^2x^2 + 1}}$

(g) $\dfrac{x + 2}{\sqrt{m^2 \pm x^2}}$

(h) $x^3\sqrt{(4 - x^2)^5}$

(i) $\dfrac{1}{\sqrt{a^2x - x^2}}$

(j) $\dfrac{x^2 + ax + b}{x^2 + 1}$

(k) $\sqrt{a^2x + x^2}$

2. Let $R(x,y)$ denote a rational function in x and y. Reduce the following integrals to integrals of rational functions.

(a) $\displaystyle\int R(x, \sqrt{ax + b})\,dx$, $a \neq 0$.

(b) $\displaystyle\int R\left(x, \sqrt[n]{\frac{ax + b}{cx + d}}\right)dx$, n an integer, $ad - bc \neq 0$.

3. Using the result of Number 2, integrate $\dfrac{x}{\sqrt{ax + b} + \sqrt{(ax + b)^3}}$.

4. Reduce to rational form $\displaystyle\int \frac{dx}{\sqrt{\frac{1 - x}{1 + x}} + \sqrt[4]{\frac{1 - x}{1 + x}}}$.

5. Express as elementary functions

(a) $\displaystyle\int \frac{dx}{\sqrt{x^2 + 1} + \sqrt{x^2 - 1}}$,

(b) $\displaystyle\int \frac{dx}{1 + \sin x}$,

(c) $\displaystyle\int \frac{dx}{1 - \cos 2x}$,

(d) $\displaystyle\int \frac{dx}{x\sqrt[4]{1 + x^4}}$,

∧(e) $\displaystyle\int \frac{dx}{\sqrt[4]{1 + x^4}}$.

6. (a) The integral $\displaystyle\int \frac{P(x)}{\sqrt{ax^2 + 2bx + c}}\,dx$, where $P(x)$ is a polynomial of degree n and $a \neq 0$ can be reduced to a rational trigonometric form as described in the text. It can be also reduced to the integration of $\dfrac{1}{\sqrt{ax^2 + 2bx + c}}$; namely for some polynomial Q of degree $(n - 1)$ and constant k .

$$\frac{P(x)}{\sqrt{ax^2 + 2bx + c}} = D\left(Q(x)\sqrt{ax^2 + 2bx + c}\right) + \frac{k}{\sqrt{ax^2 + 2bx + c}} .$$

Show how to find Q and k .

(b) Using (a), integrate $\dfrac{t^5 - t^3 + t}{\sqrt{1 - t^2}}$.

(c) Calculate the integral of (b) by using trigonometric substitutions, and compare the merits of the two methods.

7. It is stated at the beginning of this section that an integral of any rational combination of x and $\sqrt{Q(x)}$, where $Q(x) = Ax^2 + Bx + C$, can be transformed into a rational combination of circular functions. Yet Theorem 10-3a takes up the case $A \neq 0$ only. Prove the result is true if $A = 0$.

8. The proof of Theorem 10-3a does not treat the cases (i) $A < 0$, $\frac{C}{A} - \frac{B^2}{4A^2} > 0$, or (ii) $A = 0$, or (iii) $\frac{C}{A} - \frac{B^2}{4A^2} = 0$. Why not?

9. State and verify the result corresponding to Theorem 10-3a for hyperbolic substitutions.

10. Using a hyperbolic substitution similar to (8) show how to transform the integral of any rational combination of $\sinh x$ and $\cosh x$ into an integral of a rational function.

11. Integrate

    (a) $\frac{1}{\sin x}$.

    (b) $\frac{1}{\cos x}$ (by a method other than that of Example 10-3d).

10-4. Integration by Parts.

(i) The basic formula. The method of integration by parts is used to integrate certain kinds of products. The method corresponds to the formula for the derivative of a product (Theorem 4-2b).

THEOREM 10-4. If f and g are continuously differentiable over a common interval containing a and b then

$$(1)\quad \int_a^b f(x)g'(x)dx = [f(b)g(b) - f(a)g(a)] - \int_a^b f'(x)g(x)dx .$$

The theorem follows directly from Theorem 4-2b and the Fundamental Theorem.

In Leibnizian notation, for $u = f(x)$, $du = f'(x)dx$ and $v = g(x)$, $dv = g'(x)dx$ we obtain for the indefinite integral corresponding to (1),

$$(2)\qquad \int u\,dv = uv - \int v\,du .$$

Integration by means of (2) is called integration by parts.

Example 10-4a. In Section 9-3(i) we encountered the problem of integrating $\log x$ which we solved by special devices (Exercises 9-3, No. 14, Example 10-2f). Now we observe that $\log x$ has an especially simple derivative and we set $u = \log x$ and $dv = 1 \cdot dx$. For v, then, we take $v = x$. Consequently, from (2)

$$\begin{aligned}\int \log x\,dx &= x \log x - \int \frac{x}{x}\,dx \\ &= x \log x - x + C\end{aligned}$$

the formula we have already obtained.

In application, (2) is used as above for the integral of a product where the product of the integral of one factor and the derivative of the other is formally integrable.

The Leibnizian notation in (2) was introduced as a shorthand for the explicit formula. But the notation suggests that we might interpret u as a function of v, and v as the inverse function of u. This idea yields an illuminating geometrical interpretation of integration by parts. Suppose that $u = f(x)$ and $v = g(x)$ where f and g have inverses. Then we can write $u = \phi(v)$ and $v = \psi(u)$ where ϕ and ψ are inverses. (The proof is left to Exercises 10-4, No. 2). Set $u_0 = f(a)$, $u_1 = f(b)$ and $v_0 = g(a)$ $v_1 = g(b)$. We have $u_i = \phi(v_i)$ and, inversely, $v_i = \psi(u_i)$ for $i = 1 , 2$. Now suppose ϕ and ψ are increasing and nonnegative. Then, from the familiar interpretation of integral as area (see Figure 10-4) we immediately have

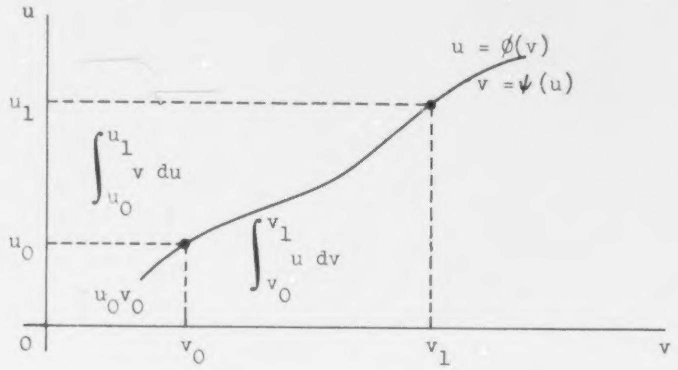

Figure 10-4

$u_1v_1 = \int_{v_0}^{v_1} u\, dv + \int_{u_0}^{u_1} v\, du + u_0v_0$, from which we at once obtain

$$\int_{v_0}^{v_1} u\, dv = [u_1v_1 - u_0u_0] - \int_{u_0}^{u_1} v\, du .$$

From the Substitution Rule we immediately recognize this equation as a form of (1) . A like geometrical argument gives the same result when ϕ and ψ are decreasing. (Compare Chapter 6, Miscellaneous Exercises, No. 12.)

In general, this interpretation of integration by parts gives the formal integral of any function which has a formally integrable inverse.

Example 10-4b. Consider

$$\int x^n \arcsin x \, dx , \qquad (n \text{ integral} , n \neq -1).$$

Since the arcsin has a simple algebraic derivative we set $u = \arcsin x$, $dv = x^n dx$ and take $v = \frac{x^{n+1}}{n + 1}$. For the domain $0 < x \leq \frac{\pi}{2}$ we have $u = \arcsin \sqrt[n+1]{(n + 1)v}$ and $v = \frac{1}{n + 1} \sin^{n+1} u$. From Theorem 10-3b we know that $\int v \, du$ can be transformed into the integral of a rational function. As we shall see (Section 10-5) rational functions are always formally integrable. It follows that $\sin^{n+1} u$ is formally integrable with respect to u and hence that $x^n \arcsin x$ is formally integrable with respect to x . Reduction to the integral of a rational function is not necessarily the most efficient way to carry out these integrations, but integration by parts can be used more effectively in other ways to execute the integrations.

The idea of Example 10-4b, for $u = f(x) dv = x^n dx$, establishes the formal integrability of $x^n f(x)$ where f is any inverse circular or hyperbolic function, and, in view of Example 10-4a, if $f(x) = \log x$.

Example 10-4c. Consider

$$\int x^r \log x \, dx , \qquad (r \text{ real}) .$$

Since $\log x$ has a simple derivative, we set $u = \log x$, $dv = x^r dx$. If $r \neq -1$ we take $v = \frac{x^{r+1}}{r + 1}$ to obtain

$$\int x^r \log x \, dx = \frac{x^{r+1}}{r + 1} \log x - \frac{1}{r + 1} \int x^r \, dx$$

$$= \frac{x^{r+1}}{r + 1} \log x - \frac{x^{r+1}}{(r + 1)^2} + C .$$

If $r = -1$, we may take $v = \log x$ to obtain

$$\int \frac{\log x}{x} dx = (\log x)^2 - \int \frac{\log x}{x} dx,$$

which yields

$$\int \frac{\log x}{x}\, dx = \frac{(\log x)^2}{2} + C ,$$

a result which is obtained more directly from the substitution $\log x = t$.

The method of Example 10-4c, for $u = f(x)$ and $dv = x^n dx$, exhibits the formal integrability of any function of the form $x^n f(x)$, when $n \neq -1$, where $f'(x)$ is any rational combination of x and $\sqrt{Q(x)}$ and $Q(x)$ is a quadratic polynomial. Integration by parts expresses the given integral in terms of the integral of $\frac{x^{n+1}}{n+1} f'(x)$ which may be transformed into the integral of a rational function by Theorem 10-3a. From the assumed integrability of rational functions, the result follows. It follows as a slight generalization that $P(x)f(x)$ is formally integrable for any polynomial function P . From this argument we observe again that if f is a logarithmic, inverse circular, or inverse hyperbolic function, then $x^n f(x)$ is formally integrable. In addition, for $h(x) = \phi(x, \sqrt{Q(x)})$, a rational combination of x and $\sqrt{Q(x)}$, the expressions $x^n \log h(x)$, $x^n \arctan h(x)$ and $x^n \operatorname{arg\,tanh} h(x)$ are all formally integrable since the derivatives of log, arctan and arg tanh are rational functions.*

Example 10-4d. Consider the integral

$$\int x\, e^x\, dx$$

whose integral we found in Example 10-2f by other means. Now we integrate by parts. Set $u = x$ $dv = e^x dx$ and $v = e^x$. Then by (2)

$$\int x\, e^x\, dx = xe^x - \int e^x\, dx$$

$$= xe^x - e^x + C$$

as we found before.

Integration by parts may be used to produce a simplification rather than a final complete integration as in Example 10-4c when $r = -1$.

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* Since arg tanh is proportional to the logarithm of a rational function it could be omitted from this list.

Example 10-4e. Consider

$$I = \int e^{bx} \sin ax \, dx$$

For $u = \sin ax$, $dv = e^{bx}\,dx$, $v = \frac{e^{bx}}{b}$, we obtain

$$I = \frac{1}{b} e^{bx} \sin ax - \frac{a}{b} \int e^{bx} \cos ax \, dx$$

$$= \frac{1}{b} e^{bx} \sin ax - \frac{a}{b} J ,$$

where

$$J = \int e^{bx} \cos ax \, dx$$

presents the same difficulties of formal integration as I. However, by the same technique, we can express J in terms of I and hopefully may obtain an equation which can be solved for I. Now take $u = \cos ax$ and $v = \frac{e^{bx}}{b}$ in (2) to obtain

$$J = \frac{1}{b} e^{bx} \cos ax + \frac{a}{b} \int e^{bx} \sin ax \, dx$$

$$= \frac{1}{b} e^{bx} \cos ax + \frac{a}{b} I .$$

Entering the expression for J above in the expression for I and solving for I, we obtain

$$I = \frac{1}{a^2 + b^2} e^{bx} (b \sin ax - a \cos ax) + C .$$

(ii) Recurrence relations. The idea here is to express an integral of the general form $\int f_n(x)\,dx$ in terms of $\int f_{n-k}(x)\,dx$.

Example 10-4f. Consider

$$I_n = \int x^r(1 - x)^n \, dx \qquad (n \geq 0 \, , \; r \neq -1).$$

Set $u = (1 - x)^n$, $dv = x^r dx$, $v = \frac{x^{r+1}}{r + 1}$. Then

$$I_n = \frac{x^{r+1}(1 - x)^n}{r + 1} + \frac{n}{r + 1} \int x^{r+1}(1 - x)^{n-1} dx$$

where, for $n = 0$, the result yields, correctly, $I_n = \frac{x^{r+1}}{r + 1}$. Now, observe that

$$x^{r+1}(1 - x)^{n-1} = -x^r[(1 - x)^n - (1 - x)^{n-1}] \; ;$$

whence,

$$I_n = \frac{x^{r+1}(1 - x)^n}{r + 1} + \frac{n}{r + 1} [I_{n-1} - I_n] \, .$$

This equation may then be solved for I_n in terms of I_{n-1} :

$$I_n = \frac{x^{r+1}(1 - x)^n}{n + r + 1} + \frac{n}{n + r + 1} I_{n-1}$$

or

$$\int x^r(1 - x)^n \, dx = \frac{x^{r+1}(1 - x)^n}{n + r + 1} + \frac{n}{n + r + 1} \int x^r(1 - x)^{n-1} \, dx \, .$$

Now this formula may be applied recursively to express I_{n-1} in terms of I_{n-2} , I_{n-2} in terms of I_{n-3} , etc., to yield

$$I_n = \frac{x^{r+1}}{n + r + 1} \left[(1 - x)^n + \frac{n(1 - x)^{n-1}}{n + r} + \frac{n(n - 1)(1 - x)^{n-2}}{(n + r)(n + r - 1)} + \dots + \frac{n(n - 1) \dots 1}{(n + r)(n + r - 1) \dots (r + 1)} \right] + C$$

Sometimes it is necessary to prepare for integration by parts by some preliminary rearrangement, as we show in the following useful example.

Example 10-4g. Consider

$$I_n = \int \cos^n x \, dx$$

We write $\cos^n x = \cos^{n-1} x \cos x$, set $u = \cos^{n-1} x$, $dv = \cos x \, dx$, $v = \sin x$, to obtain

$$I_n = \cos^{n-1} x \sin x + (n - 1) \int \cos^{n-2} x \sin^2 x \, dx$$

$$= \cos^{n-1} x \sin x + (n - 1) \int \cos^{n-2} x \, (1 - \cos^2 x) dx \, .$$

Thus,

$$I_n = \cos^{n-1} x \sin x + (n - 1)[I_{n-2} - I_n].$$

Solving for I_n, we have

$$I_n = \frac{\cos^{n-1} x \sin x}{n} + \frac{n - 1}{n} I_{n-2} \, .$$

Since the subscript is lowered by 2 at each step we observe for n even that the recursive reduction of the integral terminates at $n = 0$ with $I_0 = \int dx = x + C$, and for u odd, at $n = 1$ with $I_1 = \int \cos x \, dx = \sin x + C$.

Often the principle use of a recurrence relation is not to obtain the formal integral in terms of elementary functions (which may not be possible) but to obtain the original integral in terms of a simpler integral.

Example 10-4h. Consider

$$I_n = \int x^n e^{-x^2} \, dx \, .$$

From $u = x^{n-1}$, $dv = x e^{-x^2} dx$, $v = -\frac{1}{2} e^{-x^2}$, we obtain

$$I_n = -\frac{1}{2} x^{n-1} e^{-x^2} + \frac{(n - 1)}{2} \int x^{n-2} e^{-x^2} \, dx$$

or

$$I_n = -\frac{1}{2}x^{n-1}e^{-x^2} + \frac{n-1}{2}I_{n-2} .$$

If n is odd, the recurrence relation gives I_n in terms of elementary functions and I_1, but $I_1 = -\frac{1}{2}e^{-x^2} + C$ is elementary and I_n is formally integrable in terms of elementary functions. If n is even, then the integration of I_n is reduced to the integration of

$$I_0 = \int e^{-x^2}\,dx .$$

This integral is not elementary. However, it is well known and much used. In terms of the error function erf (the area under the normal probability curve) given by

$$\operatorname{erf} x = \frac{1}{\sqrt{2\pi}}\int_0^x e^{-\frac{t^2}{2}}\,dt$$

we have

$$I_0 = \sqrt{\pi}\,\operatorname{erf}\left(\frac{x}{\sqrt{2}}\right) + C .$$

The common tables of the error function enable us to work with it numerically just as conveniently as the circular functions.

Exercises 10-4

1. Integrate the following.

   (a) $x \sin 3x$

   (b) $x \cdot 5x$

   (c) $x^3 e^{-2x}$

   (d) $\sqrt{x} \log ax$

   (e) $\log^2 bx$

   (f) $\log^3 x$

   (g) $\arccos 7x$

   (h) arg sinh ax

   (i) arg tanh bx

   (j) arg tanh $\sqrt{bx}$

   (k) $\arctan \sqrt[3]{x}$

   (ℓ) $x \arctan x$

   (m) $\dfrac{\arccos x/m}{\sqrt{x+m}}$

   (n) $x \sin^2 x$

   (o) $x^2 \sin x$

   (p) $x^2 \arcsin ax$

   (q) $\cos^3 2x$

   (r) $\sin^5 x$

   (s) $\sin (\log ax)$

   (t) $x \tan^2 x$

   (u) $(\arcsin x)^2$

   (v) $\sin ax \cos bx$

2. Support the geometrical interpretation of integration by parts by showing for $u = f(x)$ and $v = g(x)$ where f and g have inverses, that $u = \phi(v)$ and $v = \psi(u)$ where ϕ and ψ are inverse functions.

3. Verify as alleged after Example 10-4b that the method of the example does demonstrate the reducibility of $\int x^n f(x)dx$ to the integral of a rational function if f is any inverse circular or hyperbolic function, or if f is the logarithmic function.

4. Establish recurrence relations for each of the following (in each case m and n are positive integers).

(a) $\int \sin^n x \, dx$.

(b) $\int x^m \log^n x \, dx$.

(c) $\int \sin^m x \cos^n x \, dx$.

(d) $\int x^n \text{arc tan } x \, dx$.

^(e) $\int x^n \text{arg sinh } x \, dx$.

(f) $\int x^n \text{arg tanh } x \, dx$.

(g) $\int x^n e^{ax} \, dx$.

^(h) $\int x^n \text{arc sin } x \, dx$.

(i) $\int \frac{1}{\sin^n x} \, dx$.

(j) $\int \frac{e^x}{x^n} \, dx$.

(k) $\int x^n \cos x \, dx$.

(Note the difference between n odd and n even).

10-5. Integration of Rational Functions.

The applied problems of Chapter 9 and the problems of formal integration in the preceding sections of this chapter were often recast in the form of the problem of integrating a rational function. For a rational function there always exists a formal integral in terms of elementary functions. The formal integral is obtained by reducing the rational function to a sum of a polynomial function and functions defined by the elementary forms

$$\frac{r}{(x - c)^n} \tag{1}$$

$$\frac{px + q}{[(x - a)^2 + b^2]^n}, \qquad (b > 0) . \tag{2}$$

It can be proved that such a reduction is possible, either from the Fundamental Theorem of Algebra which requires the theory of functions of a complex variable, or directly by new algebraic techniques. In either case a complete proof would take us outside the frame of this text.

The reduction of a rational function into the sum of a polynomial and terms of the form (1) and (2) is called a decomposition into partial fractions. We give one simple example.

Example 10-5a. A common case (as in Section 9-4) is given by the rational expression

$$\frac{1}{(x - a)(x - b)} = \frac{1}{b - a}\left(\frac{1}{x - b} - \frac{1}{x - a}\right), \qquad a \neq b . \tag{3}$$

From the decomposition (3) we immediately obtain the integral

$$\int \frac{dx}{(x - a)(x - b)} = \frac{1}{b - a}(\log |x - b| - \log|x - a|) + C$$

$$= \frac{1}{b - a} \log \left|\frac{x - b}{x - a}\right| + C .$$

Let R be any rational function. By long division it is always possible to put $R(x)$ in the form

$$R(x) = S(x) + \frac{P(x)}{Q(x)}$$

where S , P , Q are polynomials and the degree of P is less than that of Q . Since the polynomial S is immediately integrable, we may omit it from consideration. It follows from the Fundamental Theorem of Algebra (Intermediate Mathematics, pp. 290ff.) that every polynomial $Q(x)$ with real coefficients has a unique factorization of the form

$$(4)\quad Q(x) = A(x-c_1)^{n_1}(x-c_2)^{n_2}\dots[(x-a_1)^2+b_1^2]^{m_1}[(x-a_2)^2+b_2^2]^{m_2}\dots$$

where the c_k are the distinct real roots of Q , and $a_k \pm ib_k$, the distinct imaginary roots $(b_k > 0)$.

Now suppose that $R(x) = \frac{P(x)}{Q(x)}$ where the degree of P is less than that of Q , and that P and Q have no common factors. Then we assert that $R(x)$ is the sum of expressions of two standard forms: for each real root C , an expression of the form

$$(5)\qquad \frac{r_1}{x-c} + \frac{r_2}{(x-c)^2} + \dots + \frac{r_n}{(x-c)^n} \qquad (r_n \neq 0)$$

where n is the multiplicity of c : for each pair of conjugate imaginary roots $a \pm ib$ an expression of the form

$$(6)\qquad \frac{p_1x+q_1}{(x-a)^2+b^2} + \frac{p_2x+q_2}{[(x-a)^2+b^2]^2} + \dots + \frac{p_mx+q_m}{[(x-a)^2+b^2]^m}, \qquad (p_m^2+q_m^2 \neq 0)$$

where m is their common multiplicity. We merely use this format as a guide without proof. In each particular case it can be verified directly that the decomposition obtained is correct. Once we have obtained and verified the correctness of the partial fraction decomposition we have reduced the integration problem to that of integrating the simple form (1) and (2).

Before we embark on the problem of integration let us see what is involved in the algebraic problem of obtaining the partial fraction decomposition. The first problem is to obtain the roots of the polynomial $Q(x)$. In general the roots of a polynomial cannot be obtained from the coefficients by a formula involving only rational operations and rational powers. There are such formulas for the roots of polynomials of third and fourth degree, but these formulas are generally useless. For example, the formula for the roots of a polynomial of third degree may involve complex quantities even when all three roots are real. For computational purposes it would be sufficient to estimate the roots numerically, but it is usually easier to estimate the integral directly (see Chapter 13). Nonetheless, the method of decomposition is valuable because often the factorization of $Q(x)$ is given by the conditions of the problem (compare Section 9-4(iv) Equation (12)) and often the factorization is easily obtained.

Next, we turn our attention to the problem of obtaining the partial fraction decomposition once the denominator is given in factored form.

First we consider the problem of obtaining the partial fraction decomposition of

$$\frac{P(x)}{Q(x)} = \frac{P(x)}{(x - c_1)(x - c_2) \ldots (x - c_n)}$$

where the roots of Q are all real and simple (of multiplicity 1) and the degree of P is less than that of Q. From the foregoing, there exist constants A_k, $(k = 1 , 2 , \ldots , n)$ such that

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x - c_1} + \frac{A_2}{x - c_2} + \ldots + \frac{A_n}{x - c_n} \,. \tag{7}$$

For $x \neq c_1$ we obtain on multiplication by $(x - c_1)$

$$A_1 = \frac{P(x)(x - c_1)}{Q(x)} - S(x)(x - c_1) = T(x)$$

where $S(x)$ is the sum of all the partial fractions but the first. In a deleted neighborhood of $x = c_1$ this equation states that the expression $T(x)$ defines the constant function $T : x \longrightarrow A_1$. Therefore

$$\begin{aligned} A_1 &= \lim_{x \sim c_1} \frac{P(x)(x - c_1)}{Q(x)} \\ &= \lim_{x \sim c_1} \frac{P(x)}{(x - c_2)(x - c_3) \ldots (x - c_n)} \end{aligned}$$

whence,

$$A_1 = \frac{P(c_1)}{(c_1 - c_2)(c_1 - c_3) \ldots (c_1 - c_n)} \,. \tag{8}$$

This last expression can be written tidily if we observe that since $Q(c_1) = 0$

$$\lim_{x \sim c_1} \frac{Q(x)}{(x - c_1)} = \lim_{x \sim c_1} \frac{Q(x) - Q(c_1)}{x - c_1} = Q'(c_1) \,.$$

Thus $A_1 = \dfrac{P(c_1)}{Q'(c_1)}$. Since c_1 is simply a symbol for any one of the roots, it does not matter which for the purpose of this discussion, we have in general,

$$A_k = \frac{P(c_k)}{Q'(c_k)} \,. \tag{9}$$

Example 10-5b. We obtain the partial fraction decomposition of

$$\frac{x^2 + x - 1}{(x + 1)x(x - 1)} .$$

Here $P(x) = x^2 + x - 1$, $Q(x) = x^3 - x$, $Q'(x) = 3x^2 - 1$. The denominator has simple zeros at -1 , 0 , and 1 . From

$$\frac{P(-1)}{Q'(-1)} = \frac{-1}{2} , \frac{P(0)}{Q'(0)} = \frac{-1}{-1} , \frac{P(1)}{Q'(1)} = \frac{1}{2} ,$$

we have

$$\frac{P(x)}{Q(x)} = - \frac{1}{2(x + 1)} + \frac{1}{x} + \frac{1}{2(x - 1)} ,$$

which is easily verified to be correct.

There are general techniques for the case of multiple real roots or imaginary roots, but in such cases it is often easier to determine the decomposition by the method of equated coefficients.*

Example 10-5c. From

$$\frac{x^3 - 1}{x(x^2 + 1)^2} = \frac{r}{x} + \frac{p_1x + q_1}{x^2 + 1} + \frac{p_2x + q_2}{(x^2 + 1)^2}$$

we obtain on multiplying both sides by $x(x^2 + 1)^2$

$$\begin{aligned} x^3 - 1 &= r(x^4 + 2x^2 + 1) + p_1(x^4 + x^2) + q_1(x^3 + x) + p_2x^2 + q_2x \\ &= (r + p_1)x^4 + q_1x^3 + (2r + p_1 + p_2)x^2 + (q_1 + q_2)x + r , \end{aligned}$$

provided $x \neq 0$. Now the coefficients of like powers on the right and left must be equal (Exercises 10-5, No.3). Thus we obtain the equations

$$\begin{gathered} r + p_1 = 0 \\ q_1 = 1 \\ 2r + p_1 + p_2 = 0 \\ q_1 + q_2 = 0 \\ r = -1 , \end{gathered}$$

from which $r = -1$, $p_1 = 1$, $q_1 = 1$, $q_2 = -1$, $p_2 = 1$. This yields

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*Usually called the method of undetermined coefficients, an irritating misnomer since the conditions do determine the coefficients.

$$\frac{x^3 - 1}{x(x^2 + 1)^2} = -\frac{1}{x} + \frac{x + 1}{x^2 + 1} + \frac{x - 1}{(x^2 + 1)^2}$$

which is easily verified to be correct.

Given the partial fraction decomposition of a rational function we complete the work of formal integration by showing how to integrate the standard forms (1) and (2). For (1) the integrals are already found. If $n > 1$, we have

(10a) $$\int \frac{r}{(x - c)^n}\, dx = -\frac{r}{(n - 1)(x - c)^{n-1}} + C$$

and if $n = 1$, then

(10b) $$\int \frac{r}{x - 1}\, dx = r \log |x - 1| + C .$$

For (2) we introduce the substitution

$$(x - a) = b \tan u \qquad \left(-\frac{\pi}{2} \leq u \leq \frac{\pi}{2}\right) ,$$

where we assume $b > 0$ (compare Example 10-3b). Using $dx = \dfrac{b}{\cos^2 u}\, du$ we obtain

$$\int \frac{px + q}{[(x - a)^2 + b^2]^n}\, dx = \int \frac{p \tan u + pa + q}{b^{2n}[1 + \tan^2 u]^n} \frac{b}{\cos^2 u}\, du$$

$$= \frac{p}{b^{2n-1}} \int \cos^{2n-3} u \sin u\, du + \frac{pa + q}{b^{2n-1}} \int \cos^{2n-2} u\, du .$$

Of the last two integrals, the first is immediately formally integrable and the second is given by the recurrence relation of Example 10-4g. We leave as an exercise the problem of completing the integration and representing the formal integral in terms of x. The resulting integral is a sum of terms of the following types (plus a constant of integration),

(11a) $$\frac{Ax + B}{[(x - a)^2 + b^2]^k}$$

where k is a positive integer, $k < n$,

(11b) $$A \log [(x - a)^2 + b^2] ,$$

(11c) $$A \arctan \frac{x - a}{b} .$$

Finally, we observe that if we know the factorization of $Q(x)$ we know the form of the integral of $\frac{P(x)}{Q(x)}$ from (10) and (11). Therefore it is sufficient to differentiate this form and determine the constants by the method of equated coefficients.

Example 10-5d. Consider

$$\int \frac{x + 1}{x^2(x^2 + 4)}\, dx ,$$

The integral must be of the form

$$a \log x + \frac{b}{x} + \alpha \log (x^2 + 4) + \beta \arctan \frac{x}{2} + C .$$

The derivative of this expression is

$$\frac{a}{x} - \frac{b}{x^2} + \frac{2\alpha x}{x^2 + 4} + \frac{2\beta}{x^2 + 4} = \frac{(a + 2\alpha)x^3 + (2\beta - b)x^2 + 4ax - 4b}{x^2(x^2 + 4)}$$

Since the numerator of this expression should be $x + 1$ we have on equating coefficients

$$a + 2\alpha = 0 \;,\; 2\beta - b = 0 \;,\; 4a = 1 \;,\; -4b = 1 \;,$$

whence

$$a = \frac{1}{4} \;,\; b = -\frac{1}{4} \;,\; \alpha = -\frac{1}{8} \;,\; \beta = -\frac{1}{8} \;.$$

It is easy to verify that this yields the correct integral.

Exercises 10-5

1. Integrate the following

(a) $\frac{x + 2}{x^2 + 3x + 1}$

(b) $\frac{x^3}{x^2 + 3x - 10}$

(c) $\frac{x^3}{x^2 + 2ax + b^2}$ $(b > |a|)$

(d) $\frac{x^2 + \alpha x + \beta}{(x - a)(x - b)}$
(Consider the cases $a \neq b$ and $a = b$)

(e) $\frac{x^2}{(x - a)(x - b)(x - c)}$, $(a \neq b \neq c)$

(f) $\frac{x^3 + 1}{x^3 - 1}$

(g) $\frac{1}{x^3 + a^2}$

(h) $\frac{(x + 2)^2}{x(x - 1)^2}$

(i) $\frac{1}{x^4 - 1}$

(ℓ) $\frac{x^4}{x^4 + 1}$

(j) $\frac{x^2}{x^4 - 1}$

(m) $\frac{1}{x^6 - 1}$

(k) $\frac{1}{x^6 + x^4}$

2. Prove from Equation (3) that if

$$Q(x) = (x - a_1)(x - a_2) \dots (x - a_n) \ , \qquad \text{where}$$

$a_1 < a_2 < \dots < a_n$, then $\frac{1}{Q(x)}$ has a decomposition into partial fractions of the form

$$\frac{1}{Q(x)} = \frac{r_1}{x - a_1} + \frac{r_2}{x - a_2} + \dots + \frac{r_n}{x - a_n} \ .$$

3. Prove if

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 = b_n x^n + b_{n-1} x^{n-1} + \dots + b_0$$

for all but finitely many numbers x , that the coefficients of like powers on the right and left are equal; i.e., $a_k = b_k$ for $k = 0 , 1 , \dots , n$.

4. Verify that $\int \frac{px + q}{[(x - a)^2 + b^2]} dx$ can be expressed as the sum of terms of the forms (11a, b, c).

10-6. Definite Integrals.

The preceding sections of this chapter were devoted primarily to the problem of finding the indefinite integral of a given function. In principle, this solves the problem of evaluating any definite integral of the function. In practice, it is often desirable or necessary to evaluate a definite integral, not by formal integration, but by some other method altogether. It may be impossible to obtain an explicit representation of the indefinite integral in terms of elementary functions, yet some special symmetry may yield the value of a given definite integral effortlessly. Even if the formal expression for the indefinite integral is obtainable, the use of a symmetry condition may be a worthwhile shortcut. Often the idea of integral remains appropriate when the Riemann integral, as strictly defined, does not exist because the range or domain of the integrand may be unbounded. In these cases, we have to extend the definition of integral in a meaningful way. All these problems are treated in this section.

(i) Symmetry. Watch for symmetries; the observation that a symmetry exists often provides a direct solution to a problem or an important simplification. We have already pointed out one useful symmetry in Section 6-4.

If f is an odd function and integrable on $[-a,a]$, then

(1) $$\int_{-a}^{a} f(x)dx = 0 .$$

Example 10-6a. Consider

$$I = \int_{-\pi}^{\pi} x\, e^{x^2} \sin^4 x\, dx .$$

It is hopeless to find the indefinite integral, and it is not needed, since $I = 0$.

If f is an integrable even function on $[-a,a]$, then

(2) $$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx .$$

Example 10-6b. Consider

$$I = \int_{-x}^{x} (a_0 + a_1t + a_2t^2 + \dots + a_{2n}t^{2n})dt \ .$$

The odd powers contribute zero and for the even powers we obtain

$$\begin{aligned} I &= 2\int_{0}^{x} (a_0 + a_2t^2 + \dots + a_{2n}t^{2n} dx \\ &= 2\left(a_0x + \frac{a_2x^3}{3} + \dots + \frac{a_{2n}x^{2n+1}}{2n+1}\right) \ . \end{aligned}$$

Often an integral which exhibits no obvious symmetry can be transformed into a symmetric integral. This is specific for each case and no general rule for discovering such symmetries can be given.

Example 10-6c. Consider

$$I = \int_{-1}^{5} \sqrt[3]{x - 2}\, dx$$

Since the graph $y = \sqrt[3]{x - 2}$ has a center of symmetry at $x = 2$, we set $u = x - 2$ and find

$$I = \int_{-3}^{3} \sqrt[3]{u}\, du = 0 \ .$$

Another important symmetry of a function is periodicity.

If the function f is integrable and periodic with period p, then the integrals of f over intervals of length p are all the same; i.e.,

$$\int_{a}^{a+p} f(x)dx = \int_{b}^{b+p} f(x)dx \tag{3}$$

for all a and b.

The statement is geometrically obvious. The graph $y = f(x)$ over any interval of length p represents the complete graph in the sense that the picture of the function from a to p is identical to the picture from $a + kp$ to $a + (k + 1)p$ where k is an integer. The entire graph can be thought of as a sequence of identical pictures of width p, laid end-to-end (Figure 10-6). If a frame of width p is laid over the graph (the

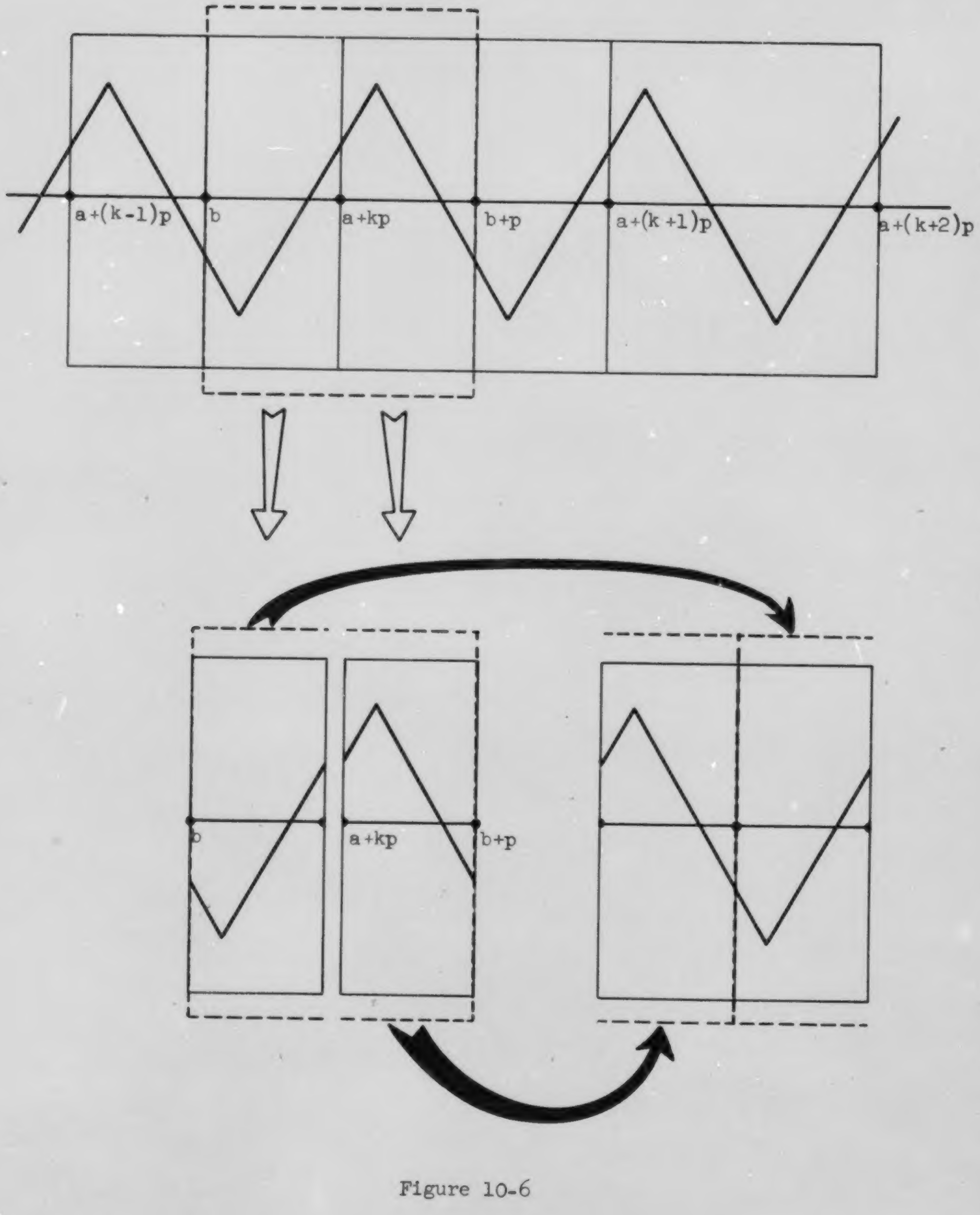
a+(k-1)p
b
a+kp
b+p
a+(k+1)p
a+(k+2)p
b
a+kp
b+p

Figure 10-6

interval $[b,b+p]$ in the figure) then the part of the total graph within the frame may be cut along a line $a + kp$ and reassembled to form the original picture by interchanging the two piece formed by the cut. This geometrical discussion is exactly paraphrased by the analytical proof. The proof is left to Exercises 10-6a, Number 12.

Example 10-6d. Consider

$$I = \int_0^{n+1/4} (a_0 + a_1 \cos 2\pi x + \dots + a_k \cos 2k\pi x)dx .$$

Since the integrand is periodic with period 1,

$$I = n\int_0^1 \sum_{\nu=0}^{k} a_\nu \cos 2\nu\pi x \, dx + \int_0^{1/4} \sum_{\nu=0}^{k} a_\nu \cos 2\nu\pi x \, dx .$$

For $\nu > 0$,

$$\int_0^1 \cos 2\nu\pi x \, dx = \frac{\sin 2\nu\pi x}{2\nu\pi}\Big|_0^1 = 0$$

and

$$\int_0^{1/4} \cos 2\nu\pi x \, dx = \frac{\sin\left(\frac{\nu\pi}{2}\right)}{2\nu\pi} .$$

Consequently,

$$I = (n + \tfrac{1}{4})\, a_0 + \frac{a_1}{2\pi} - \frac{a_3}{6\pi} + \frac{a_5}{10\pi} - \dots .$$

(ii) Special reductions. The general form of a recurrence relation for a definite integral is

$$\int_a^b f_n(x)dx = g_n(x)\Big|_a^b + c_n\int_a^b f_{n-1}(x)dx .$$

Quite often specific problems lead to integrals for which the "boundary" term

$$g_n(x)\Big|_a^b = g_n(b) - g_n(a) ,$$

is zero for $n > 0$, say. If so, we immediately have

$$\int_a^b f_n(x) = c_n\, c_{n-1} \cdots c_1 \int_a^b f_0(x) \ .$$

Thus in Example 10-4f, we could conclude at once from

$$\int x^m(1 - x)^n \, dx = \frac{x^{m+1}(1 - x)^n}{m + n + 1} + \frac{n}{n + m + 1} \int x^m(1 - x)^{n-1} \, dx$$

that

$$\int_0^1 x^m(1 - x)^n \, dx = \frac{n(n - 1) \ldots 1}{(n + m + 1)(n + m) \ldots (m + 2)} \int_0^1 x^m \, dx$$

$$= \frac{n(n - 1) \ldots 1}{(n + m + 1)(n + m) \ldots (m + 1)} \ .$$

Thus we obtain an important connection with the binomial coefficients:

$$\int_0^1 x^m(1 - x)^n \, dx = \left[(n + m + 1) \binom{n+m}{m} \right]^{-1} \ .$$

Example 10-6e. A case of special interest is

$$I_\nu = \int_0^{\pi/2} \cos^n x \, dx \ .$$

From the result of Example 10-4g, we have

$$I_\nu = \frac{\cos^{\nu-1} x \sin x}{\nu} \Big|_0^{\pi/2} + \frac{\nu - 1}{\nu} I_{\nu-2} \ .$$

For $\nu > 1$, this yields simply

(4) $$I_\nu = \frac{\nu - 1}{\nu} I_{\nu-2} \ .$$

For ν even, $\nu = 2n$, we obtain

(5a) $$I_{2n} = \frac{(2n - 1)(2n - 3) \ldots 1}{2n(2n - 2) \ldots 2} \frac{\pi}{2} \ .$$

For ν odd, $\nu = 2n + 1$, we obtain

(5b) $$I_{2n+1} = \frac{2n(2n - 2) \ldots 2}{(2n + 1)(2n - 1) \ldots 3} \ .$$

From (5a) and (5b) there can be obtained a graceful representation of $\frac{\pi}{2}$ known as Wallis's Product.[*] Observe that

$$\frac{\pi}{2} = \frac{2^2}{1\cdot 3}\cdot\frac{4^2}{3\cdot 5}\cdot\frac{6^2}{5\cdot 7}\cdots\frac{(2n)^2}{(2n-1)(2n+1)}\,\frac{I_{2n}}{I_{2n+1}}\ .$$

Now, since $0 \leq \cos x \leq 1$ on $[0,\frac{\pi}{2}]$ we have $\cos^{\nu+1} x \leq \cos^{\nu} x$ for all ν so that $I_{\nu+1} \leq I_{\nu}$. It follows that $I_{2n+1} \leq I_{2n} \leq I_{2n-1}$, and since $I_{2n-1} = \frac{2n+1}{2n} I_{2n+1}$, that

$$1 \leq \frac{I_{2n}}{I_{2n+1}} \leq 1 + \frac{1}{2n}\ .$$

From the Squeeze Theorem (Theorem 3-4f, Corollary 2) which is easily extended to this kind of limit (the epsilonic proof is an exact parallel), we obtain $\lim\limits_{n\sim\infty} \frac{I_{2n}}{I_{2n+1}} = 1$, whence

$$\frac{\pi}{2} = \frac{2^2}{1\cdot 3}\cdot\frac{4^2}{3\cdot 5}\cdot\frac{6^2}{5\cdot 7}\cdots$$

where by this infinite product, we mean simply

$$\lim_{n\sim\infty}\left[\frac{2^2}{1\cdot 3}\cdot\frac{4^2}{3\cdot 5}\cdot\frac{6^2}{5\cdot 7}\cdots\frac{(2n)^2}{(2n-1)(2n+1)}\right]$$

$$= \lim_{n\sim\infty}\frac{1}{2n+1}\left[\frac{2^{2n}(n!)^2}{(2n)!}\right]^2$$

The verification that the two expressions in these limits are equal is left as an exercise.

Wallis's Product is not useful for estimating π , but it will be used (Chapter 13) to obtain Stirling's asymptotic formula for n! .

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[*] John Wallis (1616 - 1703), English.

Exercises 10-6a

Evaluate the following definite integrals:

1. $$\int_{-99}^{99} \frac{\sin^{99} \frac{x}{99}}{x^2 + (99)^2}\, dx .$$

2. $$\int_0^1 x^3 e^{-3x^2}\, dx .$$

3. $$\int_1^e \log^3 x\, dx .$$

4. $$\int_0^{\pi/2} \sin^m x\, dx ,$$ (m, a positive integer).

5. $$\int_0^{\pi/2} \sin^m x \cos^m x\, dx ,$$ (m, a positive integer).

6. $$\int_0^{\pi/2} \frac{dx}{a + b \cos x} \quad a > b \geq 0 .$$

7. $$\int_0^{\pi/2} \sin^7 x \cos^3 x\, dx .$$

8. $$\int_1^2 \frac{dx}{x + x^5} .$$

9. $$\int_0^b \sqrt{b^2 - x^2}\, dx .$$

10. $$\int_{-\pi/4}^{\pi/4} \frac{\sin^5 \theta + 1}{a^2 \sin^2 \theta + b^2 \cos^2 \theta}\, d\theta ,$$ $a > 0$, $b > 0$.

11. Compare $\int_0^{-a} f(x)dx$ with $\int_{-a}^{0} f(x)dx$ when f is even or odd to derive the results (1) and (2) of the text by a method other than the one you employed for Exercises 6-4, Number 4.

12. Prove if f is integrable and periodic of period p , then for all a and b

$$\int_a^{a+p} f(x)dx = \int_b^{b+p} f(x)dx .$$

∧13. Prove that if $n \geq 2$ then

$$.500 < \int_0^{1/2} \frac{dt}{\sqrt{1 - t^n}} < .524 .$$

∧14. Prove that $$\int_{-\pi}^{\pi} \frac{2x(1 + \sin x)}{1 + \cos^2 x}\, dx = \pi^2 .$$

15. Show $$\frac{2^2}{1 \cdot 3} \cdot \frac{4^2}{3 \cdot 5} \cdot \frac{6^2}{5 \cdot 7} \cdots \frac{(2n)^2}{(2n - 1)(2n + 1)} = \frac{1}{2n + 1}\left[\frac{2^{2n}(n!)^2}{(2n)!}\right]^2 .$$

16. Determine the value exact to two decimal places of

$$\int_1^{e^{36.1}} \frac{\sin(\pi \log x)}{x}\, dx \; .$$

17. Evaluate

$$\int_{-\pi/4}^{\pi/4} \frac{t + \frac{\pi}{4}}{2 - \cos 2t}\, dt \; .$$

(Hint: Express the integrand as the sum of a symmetric part and an integrable part.)

(iii) *Improper integrals*. Often a problem requires the evaluation of a definite integral over an interval where the integrand may be discontinuous or undefined at isolated points. For example, in Section 9-3(i) we sought to evaluate $I = \int_0^{N_0} \log N \, dN$. Although $\log N$ is not defined at $N = 0$ and is even unbounded in any neighborhood of 0 we found it perfectly reasonable that the integral should represent a definite number $I = N_0 \log N_0 - N_0$ (Exercises 9-3, No. 14). We gave the symbol I a numerical value, but in so doing we defined something new. On two scores, we cannot describe I as a Riemann integral: the integrand is not defined and it is not bounded on the interval $[0, N_0]$ of integration. Since $\int_x^{N_0} \log N \, dN$ exists for $0 < x \leq N_0$ it is appropriate to define the integral from 0 to N_0 by

$$I = \lim_{x \to 0} \int_x^{N_0} \log N \, dN .$$

More generally, let f be any function which has discontinuities on the interval $[a,b]$. We say that the integral of f over $[a,b]$ is *improper*. We shall interpret such an improper integral as a limit, as in the cited example, provided the requisite limit exists. For this purpose we use the idea of right- and left-sided limits (cf. Exercises 3-4, No. 16). Let ϕ be defined on a domain which contains the open interval (a,b). We consider the function ϕ^* which is the restriction of the function ϕ to (a,b) and define the *right-sided limit* of F at a as

$$\lim_{x \to a^+} \phi(x) = \lim_{x \to a} \phi^*(x)$$

and the *left-sided limit* of ϕ at b, similarly, as

$$\lim_{x \to b^-} \phi(x) = \lim_{x \to b} \phi^*(x) .$$

Now, let f be integrable over any closed subinterval of (a,b) and let ξ be any point of (a,b). We introduce the function ϕ defined on (a,b) by

$$\phi(x) = \int_\xi^x f(t)dt .$$

We define the generalized integral of f over $[a,b]$ to be

(1)
$$I = \lim_{\beta \sim b^-} \phi(\beta) - \lim_{\alpha \sim a^+} \phi(\alpha)$$
$$= \lim_{\alpha \sim a^+} \int_\alpha^\xi f(t)dt + \lim_{\beta \sim b^-} \int_\xi^\beta f(t)dt ,$$

provided the limits exist, and the integrals in (1) are defined in the sense of Section 6-3.

This new definition includes the Riemann integrals defined earlier (Exercises 10-6b, No. 1a) and extends the concept to include cases not covered by Definition 6-3.

Example 10-6f. Consider the arclength L of the upper half of the unit circle $y = \sqrt{1 - x^2}$ for $-1 \leq x \leq 1$. From the definition of arclength of Section 6-3(iv),

$$L = \int_{-1}^{1} \frac{1}{\sqrt{1 - x^2}} dx .$$

This is an improper integral; the integrand is discontinuous at both ends and it is unbounded. To evaluate the integral we apply the basic integration Formula (8) of Table 10-1a and obtain

$$L = \lim_{\beta \sim 1^-} \arcsin \beta - \lim_{\alpha \sim -1^+} \arcsin \alpha .$$

Since arcsin is the continuous inverse of sin where the domain of sin is restricted to $[-1,1]$,

$$L = \frac{\pi}{2} - (-\frac{\pi}{2}) = \pi ,$$

as we expect from geometry.*

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*This argument may appear to assume what is to be proved since sin and arcsin were introduced geometrically. However, in Section 8-5 and Appendix 8 these functions were defined purely analytically. Since sin is everywhere defined and continuous, and since arcsin is increasing on the open interval $(-1,1)$, it follows that arcsin is defined and continuous on the closed interval $[-1,1]$. From the definition of π, Formula (9) of Appendix 8,

$$\frac{\pi}{4} = \arcsin \frac{1}{\sqrt{2}} ,$$

it was shown with the aid of the addition theorems that

$$\frac{\pi}{2} = \arcsin 1 .$$

Since sin is an odd function (Exercises 8-5, No. 11) it follows that $\arcsin(-1) = \frac{-\pi}{2}$ and the argument is completed. In this way we finally establish the connection between the analytical and geometrical conceptions of the circular functions.

In Exercises 9-3, Number 14, by a geometrical argument, it was suggested that the evaluation of the improper integral $\int_0^{N_0} \log N \, dx$ could be accomplished in terms of another kind of improper integral,

$$I = \int_{-\infty}^{\log N_0} e^x dx .$$

This integral is naturally defined by

$$I = \lim_{\alpha \sim -\infty} \int_{\alpha}^{\log N_0} e^x dx$$

and is easily evaluated.

Example 10-6g. Consider

$$I = \int_0^{\infty} x^n e^{-x} dx .$$

It is not obvious that the indicated limit exists but we may obtain the indefinite integral $J_n = \int x^n e^{-x} dx$ using integration by parts and explore the question of existence afterward. Setting $u = x^n$, $dv = e^{-x}dx$ and integrating by parts we obtain the recurrence relation

$$J_n = -x^n e^{-x} + nJ_{n-1},$$

whence, $J_n = \phi(x) + C$, where

$$\phi(x) = -e^{-x}[x^n + nx^{n-1} + n(n-1)x^{n-2} + \ldots + n!] .$$

Now, by Lemma 8-3,

$$\lim_{\beta \sim \infty} e^{-\beta} x^k = 0 .$$

Consequently,

$$I = \lim_{\beta \sim \infty} \phi(\beta) - \lim_{\alpha \sim 0} \phi(\alpha) = n! .$$

Thus we obtain the representation

$$n! = \int_0^{\infty} x^n e^{-x} \, dx ,$$

interesting because it suggests the possibility of extending the definition of the factorial function from the domain of nonnegative integers in a simple way of a function on the domain of all nonnegative real numbers.

With these examples in mind we extend the definition of integral as follows.

DEFINITION 10-6. Consider an interval (a,b), a partition $\{x_0, x_1, \ldots, x_n\}$ and a function f integrable over every closed subinterval of (a,b) which contains no points of the partition. Here we also include the possibilities that $a = x_0 = -\infty$ or that $b = x_n = \infty$.

The integral (in the extended sense) of f over $[a,b]$ is defined to be

$$\int_a^b f(t)dt = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f(t)dt, \tag{2}$$

where each of the terms of the sum is defined by (1), and each of the indicated limits exists. To complete the definition we define

$$\int_a^a f(t)dt = 0$$

and if $b < a$,

$$\int_a^b f(t)dt = -\int_b^a f(t)dt,$$

provided $\int_b^a f(t)dt$ exists. (Compare Definitions 6-4a, b.)

If $a = -\infty$, or $b = \infty$, or any of the partition points x_i is a point of discontinuity of f we say the integral is improper.

The basic theorems for Riemann integrals also hold for integrals in the extended sense:

(a) If f and g are integrable over $[a,b]$ in the extended sense, and $f(x) \leq g(x)$, then

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx$$

(Theorem 6-4a).

(b) If a, b and c are points of an interval over which f is integrable in the extended sense, then

$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$$

(Corollary to Theorem 6-4b).

(c) If f and g are integrable over $[a,b]$ in the extended sense, then

$$\int_a^b [\alpha f(x) + \beta g(x)] = \alpha\int_a^b f(x)dx + \beta\int_a^b g(x)dx$$

(Theorem 6-4c).

(d) Let f be continuous on (a,b) and integrable over $[a,b]$. On the domain consisting of an open interval I with endpoints α and β (we permit $\alpha > \beta$), let g be continuously differentiable. If the range of g is in (a,b), and $\lim\limits_{t\to\alpha} g(t) = a$ and $\lim\limits_{t\to\beta} g(t) = b$, then

$$\int_\alpha^\beta fg(t)\, g'(t)dt = \int_a^b f(x)dx .$$

(Substitution Rule, Theorem 10-2.)

The proofs of (a)-(d) are left to Exercises 10-6b, Number 2.

In general, we write the symbol $\int_a^b f(x)$ first and question its significance later. If each of the indicated limits in (2) exists we say that the integral converges or that it is a convergent integral. If any one of the limits fails to exist, the improper integral is called divergent. It should be kept in mind that a divergent integral is not a number; it is a meaningless symbol, and operations with meaningless symbols are likely to lead to meaningless results. For example, $\int_{-1}^{1} \frac{dx}{x^2} = -2$.

Only after it is proved that an improper integral is convergent can we rely on the results of computations in which the integral is involved. We need criteria to determine whether an improper integral converges or diverges. One of the most broadly useful criteria is comparison with a nonnegative test function for which the integral is known either to converge or diverge.

THEOREM 10-6a. Let f be Riemann integrable over every closed subinterval of (a,b). If $|f(x)| \leq g(x)$ and $\int_a^b g(x)dx$ coverges, then $\int_a^b f(x)$ converges.

Corollary. Let h be Riemann integrable on every closed subinterval of (a,b). If $f(x) \geq h(x) \geq 0$ and $\int_a^b h(x)dx$ diverges, then $\int_a^b f(x)$ diverges.

The proof of the theorem is given in Section A10.

Since f is Riemann integrable in any closed subinterval of (a,b), the comparison between f and the test function g may be restricted to any one-sided neighborhoods of the endpoints. (See the proof of Theorem A10a.) One of the most useful test functions is the power function

$$g : x \longrightarrow \frac{A}{(x - a)^{\alpha}} .$$

THEOREM 10-6b. The integral

$$\int_a^b \frac{dx}{|x - a|^p}$$

converges if $p < 1$ and diverges if $p \geq 1$.

We prove the case for $a < b$.

Proof. For $a < \alpha < b$,

$$\int_{\alpha}^b \frac{dv}{(x - a)^p} = \begin{cases} \frac{1}{1 - p}[(b - a)^{1-p} - (\alpha - a)^{1-p}] & \text{for } p \neq 1 , \\ \log (b - a) - \log (\alpha - a) & \text{for } p = 1 . \end{cases}$$

Also, $\lim_{\alpha \to a^+} (\alpha - a)^{1-p}$ exists if $p < 1$ and does not exist if $p > 1$, and $\lim_{\alpha \to a^+} \log(\alpha - a)$ does not exist. Thus, since

$$\int_a^b \frac{dx}{(x - a)^p} = \lim_{\alpha \to a^+} \int_\alpha^b \frac{dx}{(x - a)^p}$$

the theorem is proved.

Example 10-6h. Consider again

$$I = \int_{-1}^{1} \frac{1}{\sqrt{1 - x^2}}\, dx$$

(Example 10-6f). For $-1 < x \leq 0$ we have $1 - x \geq 1$, thus $\frac{1}{\sqrt{1 - x^2}} \leq \frac{1}{(x + 1)^{1/2}}$ and $\int_{-1}^{0} \frac{1}{\sqrt{1 - x^2}}\, dx$ converges. Similarly, $\int_{0}^{1} \frac{1}{\sqrt{1 - x^2}}\, dx$ can be shown to converge, and we conclude that I converges by Theorem 10-6b.

Implicit in Theorem 10-6b are the conditions $a \neq -\infty$ and $b \neq \infty$, otherwise the test functions $\frac{A}{(x - a)^\alpha}$ and $\frac{B}{(b - x)^\beta}$ would not be defined. We need similar criteria for unbounded intervals.

THEOREM 10-6c. The integral

$$\int_a^\infty \frac{dx}{x^p}$$

converges if $p > 1$ and diverges if $p \leq 1$.

The proof is left as an exercise (Exercises 10-6b, No. 6).

Example 10-6i. Consider

$$\int_{-\infty}^{\infty} e^{-x^2/2}\, dx .$$

For $u \geq 0$ we have from Section 8-6, (3), $e^u \geq 1 + u > u$; consequently, $e^{-u} = \frac{1}{e^u} < \frac{1}{u}$. It follows that

$$e^{-x^2/2} < \frac{2}{x^2}$$

for all x . Taking $f(x) = e^{-x^2/2}$ and $g(x) = \frac{2}{x^2}$ we conclude from

Theorem 10-6c that $\int_1^\infty e^{-x^2/2}\,dx$ and $\int_{-\infty}^{-1} e^{-x^2/2}\,dx$ converge. Hence

$$\int_{-\infty}^{\infty} e^{-x^2/2}\,dx = \int_{-\infty}^{-1} e^{-x^2/2}\,dx + \int_{-1}^{1} e^{-x^2/2}\,dx + \int_1^\infty e^{-x^2/2}dx$$

converge. The indefinite integral of $e^{-x^2/2}$ cannot be given in terms of elementary functions, but the integral over $[-\infty,\infty]$ can be computed in other ways and is known to be $\sqrt{2\pi}$.

More refined tests than comparison with power functions may be necessary on occasion.

Example 10-6j. Consider

$$I = \int_0^{\pi/2} \log \sin x\,dx \text{ .}$$

The difficulty here occurs at $x = 0$. However, since $\sin x$ behaves like x near $x = 0$ and we have evidence that $\int_0^a \log x\,dx$ converges (Exercises 9-3, No. 14), we have reason to believe that I converges. For the proof we observe for x in $[0,\frac{\pi}{2}]$ that

$$\sin x \geq \frac{2x}{\pi}$$

(see Exercises 10-6b, No. 13). Since $\log u$ is negative for $u < 1$ and $\log$ is an increasing function, we have

$$|\log \sin x| \leq -\log \frac{2x}{\pi} = \log \frac{\pi}{2} - \log x \text{ .}$$

Now, integration by parts yields

$$-\int \log x\,dx = x(1 - \log x) + C \text{ ,}$$

and

$$\begin{aligned} -\lim_{x \sim 0} x \log x &= \lim_{x \sim 0} x \log \frac{1}{x} \\ &= \lim_{z \sim \infty} \frac{\log z}{z} \\ &= 0 \end{aligned}$$

as we know from Lemma 8-3. Thus, taking $g(x) = -\log \frac{2x}{\pi}$ as our test function and applying Theorem 10-6b, we establish the convergence of I .

Exercises 10-6b

1. (a) Let f be Riemann integrable over $[a,b]$. Show that the integral of f in the sense of (1) exists and is equal to the Riemann integral.

   (b) Show that if f is continuous on (a,b) and the limits $\lim\limits_{x \sim a^+} f(x)$ and $\lim\limits_{x \sim b^-} f(x)$, exist, then $\int_a^b f(x)dx$ exists in the sense of (1).

2. Show that the basic theorems for Riemann integrals hold also for integrals in the extended sense:

   (a) If f and g are integrable over $[a,b]$ in the extended $f(x) \leq g(x)$, then

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx$$

   (Theorem 6-4a).

   (b) If a, b and c are points of an interval over which f is integrable in the extended sense, then

$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$$

   (Corollary to Theorem 6-4b).

   (c) If f and g are integrable over $[a,b]$ in the extended sense, then

$$\int_a^b [\alpha f(x) + \beta g(x)] = \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx$$

   (Theorem 6-4c).

   (d) Let f be continuous on (a,b) and integrable over $[a,b]$. On the domain consisting of an open interval I with endpoints α and β (we permit $\alpha > \beta$), let g be continuously differentiable. If the range of g is in (a,b), and $\lim\limits_{t \sim \alpha} g(t) = a$ and $\lim\limits_{t \sim \beta} g(t) = b$, then

$$\int_\alpha^\beta fg(t)\ g'(t)dt = \int_a^b f(x)dx$$

   (Substitution Rule, Theorem 10-2). The proof requires the demonstration of existence for the integral on the left.

For this purpose it is convenient to introduce the concepts "neighborhood of infinity" and "neighborhood of minus infinity." A neighborhood of ∞ is an open ray of the form $\{x : x > a\}$. Similarly, a neighborhood of $-\infty$ is a ray of the form $\{x : x < a\}$. For a neighborhood of ∞ or $-\infty$, the neighborhood and the deleted neighborhood are the same. Furthermore a neighborhood of ∞ is a left-sided neighborhood; a neighborhood of $-\infty$ is a right-sided neighborhood. We now extend the meaning of

$$\lim_{x \sim a} f(x) = b ,$$

so that a and b may not be real numbers but ∞, or $-\infty$; that is, every deleted neighborhood of a contains points of the domain of f and for each neighborhood J of b there exists a deleted neighborhood I of a wherein f maps the points of its domain into J.

3. Prove the corollary to Theorem 10-6a.

4. Prove Theorem 10-6b when $a > b$.

5. Prove that $\int_0^a \frac{A}{|x - a|^p} dx$ converges if $p < 1$ and diverges if $p \geq 1$.

6. Prove Theorem 10-6c.

7. Test for convergence and divergence,

$$I = \int_0^\infty x^r \, dx .$$

8. Evaluate those of the following improper integrals which converge.

(a) $\int_{-2}^{4} \frac{dx}{\sqrt[5]{(4 - x)^3}}$

(d) $\int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - \cos\theta}}$

(b) $\int_0^{a^{1/m}} \frac{x^{m-1}dx}{\sqrt{a^2 - x^{2m}}}$, $m > 0$.

(e) $\int_0^\infty (ax + b)^2 e^{-x} dx$

(c) $\int_0^1 \log^2 x \, dx$

(f) $\int_1^\infty x^3 e^{-2x^2} dx$

(g) $\int_0^\infty x^{n-1} e^{-ax}\, dx$; $a > 0$; n , natural number.

(h) $\int_0^\infty \frac{dx}{x\sqrt{1 + x^3}}$

(i) $\int_1^\infty \sqrt[3]{\frac{x - 1}{x + 1}} \frac{dx}{x}$

(j) $\int_a^b \frac{dx}{\sqrt{(b - x)(x - a)}}$, $a < b$.

9. Determine whether or not the following integrals are convergent and evaluate when practical.

(a) $\int_1^\infty \frac{x^3 - 1}{x^4 + 1}\, dx$

(b) $\int_1^\infty \frac{\log(x^2 + 1)}{x^{3/2}}\, dx$

(c) $\int_1^e \frac{dt}{t\sqrt{\log t}}$

(d) $\int_0^2 \frac{dt}{e^{\sqrt{t}} - 1}$

(e) $\int_0^1 \log^n x\, dx$

(f) $\int_0^\infty e^{-u} \log^n u\, du$

(g) $\int_0^\infty \frac{x^2 - 1}{(x^2 + 1)\sqrt{1 + x^4}}\, dx$

(h) $\int_{-1}^\infty \frac{x^2 dx}{\sqrt{|x^7 + x^3 + 1|}}$

(i) $\int_0^\infty \frac{dx}{\sqrt{2x^4 - 2x^3 - x^2 + 1}}$

(j) $\int_0^1 \frac{dx}{\sqrt{1 - x^4}}$

(k) $\int_0^\pi \frac{\sin x}{\sqrt{1 - \cos x}}\, dx$

(ℓ) $\int_0^\pi \frac{\cos x}{\sqrt{1 - \cos x}}\, dx$

10. Consider

$$I = \int_{-\infty}^{\infty} \sqrt{|R(x)|}\, dx$$

where $R(x)$ is a rational function, $R(x) = \frac{P(x)}{Q(x)}$, where the polynomials $P(x)$ and $Q(x)$ have no common factors and $Q(x)$ has only simple roots. Determine the conditions under which I converges or diverges.

11. Test for convergence.

(a) $\displaystyle\int_1^e \frac{dx}{x^m \log x}$ (c) $\displaystyle\int_e^\infty \frac{dx}{x^m \log x}$

(b) $\displaystyle\int_1^e \frac{dx}{x(\log x)^m}$ (d) $\displaystyle\int_e^\infty \frac{dx}{x(\log x)^m}$

12. Using estimates in the manner of Example 10-6i, obtain an upper estimate of

$$\int_\infty^\infty e^{-x^2/2}\,dx \,.$$

13. Complete the demonstration of Example 10-6j that $\displaystyle\int_0^{\pi/2} \log \sin x \, dx$ converges by proving

$$\sin x \geq \frac{2x}{\pi} \qquad 0 \leq x \leq \frac{\pi}{2} \,.$$

(Hint: sin x is flexed downward on the given interval).

14. Give an example of a function f which is Riemann integrable over every subinterval [a,b] of (0,1) and for which the integrals are bounded,

$$\left|\int_a^b f(x)\,dx\right| \leq M \,,$$

yet which is not itself integrable over [0,1] in the extended sense.

15. Prove that if f is continuous on (a,b) and if $\displaystyle\int_a^b |f(x)|\,dx$ exists, then $\displaystyle\int_a^b f(x)\,dx$ exists.

16. Show that if the generalized integral defined by (1) exists, then it is independent of the choice of ξ .

17. Verify that the value of the integral in the extended sense is not affected if additional points are included in the partition used in Definition 10-6.

18. A function f is said to be *piecewise continuous* on [a,b] if there is a partition $\{x_0,x_1,x_2,\ldots,x_n\}$ of the interval such that f is continuous on each open subinterval (x_{i-1},x_i) and the one-sided limits, $\lim\limits_{x\to x_{i-1}} f(x)$, $\lim\limits_{x\to x_i}{}_{+} f(x)$, exist $(i = 1,2,3,\ldots,n)$. Show that f is integrable in the extended sense.

10-7. Linear Differential Equations of First Order.

The theory of differential equations is rich, deep and fascinating, with ramifications spreading far into mathematics and the sciences. The calculus lies at the beginning of this theory. A differential equation defines a class of functions, its solutions. We adopted this point of view in Section 8-5 to define the circular functions by means of the differential equation $D^2u + u = 0$. The solutions of differential equations form a far broader class of functions than those encountered so far. In this section and the next, we shall consider only such equations as may be solved in terms of functions we know already, elementary functions and their integrals. This is a serious and artificial limitation. Still, the few basic types of differential equations we shall study, for all their simplicity, are quite versatile in application to the sciences and mathematics.

The principal concern of this chapter is the integration of the simplest differential equation,

$$(1) \qquad Du = f$$

where f is given, and the function u is to be determined. In Chapter 9, although the equations had diverse scientific origins, for the most part they were of the simple form

$$(2) \qquad Du = a + bu + cu^2$$

where a, b and c are constants, with various interpretations of function u and the constants a, b, c. It was this basic similarity of mathematical structure which served as the unifying thread of that chapter. In this section we consider the differential equation

$$(3) \qquad Du + p \cdot u = f \text{ , or } \frac{dy}{dx} + p(x)y = f(x) \text{ ,}$$
$$\text{for all } x \text{ in the domain of } u \text{ ,}$$

where $y = u(x)$ and p and f are given continuous functions. This class of equations includes the type (1) $\big(p(x) = 0\big)$, but includes only those equations of type (2) for which $c = 0$, with constant coefficients $f(x) = a$, $p(x) = b$ (see Exercises 10-7, No. 2c).

It is convenient to introduce the idea of a differential operator which maps a differentiable function onto another function (thus a differential operator can be considered as a function on the domain of differentiable functions). For our present purposes, we need not define the concept of differential operator in all generality. We merely point out that for the

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differential operator given by

(3a) $$L = D + p : \phi \longrightarrow D\phi + p \cdot \phi ,$$

Equation (3) can be written in the form

(3b) $$L[u] = f .$$

For example, the operator $L[u] = Du + p \cdot u$ where $p(x) = \sin x$ takes $u(x) = x^2$ into $L[x^2] = 2x + x^2 \sin x$. Hence $u(x) = x^2$ is a solution of $L[u] = 2x + x^2 \sin x$. The problem posed by Equation (3) or (3b) is to find those functions u which have f as their image under L. The operator L of (3a) and the Equation (3b) are called linear since for any linear combination of functions ϕ and ψ in the domain of L (differentiable functions),

$$L[\alpha\phi + \beta\psi] = \alpha L[\phi] + \beta L[\psi] .$$

The operator and equation are said to be of first order since the expression for $L[\phi]$ involves no derivatives of ϕ of order higher than the first. The function f in (3b) is called the forcing term (from physical applications). If the forcing term is zero the equation is said to be homogeneous. The homogeneous equation $L[u] = 0$ is called the reduced equation of $L[u] = f$.

(i) The reduced equation. The solution of (3) begins with the reduced equation

(4) $$Du + p \cdot u = 0 .$$

This equation has the solution $u = 0$, the so-called trivial solution. Now suppose there is a nontrivial solution u of (4), which is nonzero at the point x_0; i.e., $y_0 = u(x_0) \neq 0$. Since u is continuous[*] it must be bounded away from zero in some neighborhood of x_0 (Lemma 3-4). Wherever y is not zero we may divide by y, hence on some interval a nontrivial solution of (4) must satisfy

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}(\log|y|) = -p(x) .$$

On integrating from x_0 to x, we find

$$\log \frac{y}{y_0} = - \int_{x_0}^{x} p(t)dt ;$$

---

[*] To satisfy the differential equation, u must even be differentiable.

thence,

$$y = u(x) = y_0 \exp\left\{-\int_{x_0}^{x} p(t)dt\right\} . \tag{5}$$

Formula (5) tells us what form a nontrivial solution of the reduced equation must have if one exists in a neighborhood of x_0. Now we observe that (5) defines a function $u : x \longrightarrow y$ on the domain of the given continuous function p and that the function u so-defined is a solution of the differential equation on this domain:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left[y_0 \exp\left\{-\int_{x_0}^{x} p(t)dt\right\}\right] \\ &= \left[y_0 \exp\left\{-\int_{x_0}^{x} p(t)dt\right\}\right]\left(-p(x)\right) \\ &= -p(x)y .\end{aligned}$$

In particular, the initial value problem for the equation of unregulated growth or decay,

$$y' = cy \quad , \quad y = y_0 \text{ at } x = 0 ,$$

where c is constant, has the unique solution

$$y = y_0e^{cx}$$

as we proved before in Theorem 8-5a.

(ii) The initial value problem for the reduced equation. In applications, Equation (4) typically describes the variation of some quantity with time. In Sections 9-2, 3 with a constant function $p : x \longrightarrow a$, for example, the euqation was used to describe processes of growth and decay. For phenomena involving time variation one of the most significant questions is to determine and describe the future states of a system in terms of a present initial state. For phenomena governed by the differential equation (4), this question takes the form of the initial value problem: given the initial state y_0 of the system at time x_0, what is the state y of the system at any later time $x > x_0$?

The association of a mathematical initial value problem with a physical problem leads to mathematical criteria of significance: the initial-value problem is said to be well-posed if it satisfies the following conditions,

(a) A solution exists (there is a future).

(b) The solution is unique (future states are determined by the present state).

To these conditions it is usually appropriate to add a third.

(c) The solution depends continuously on the initial data (minuscule causes should not produce immediate enormous effects).

We do not emphasize condition (c) here since it will follow from the explicit solutions of the equations we shall treat.

It is quite reasonable to look backward in time and inquire about the earlier behavior of a system which produced a present state (e.g., Exercises 9-3, No. 12). For the purposes of mathematical analysis it makes no significant difference whether we are seeking a forward or backward solution of a differential equation; that is, the parameter for time may be denoted by x or $-x$, indifferently. The mathematical concept of initial value problem includes all problems in which an appropriately described state of a system is given for one value of a parameter--not always time--and the variation of the state of the system in some parameter interval containing the given value is to be determined.

The initial value problem

$$\begin{cases} L[u] = Du + p \cdot u = 0 \\ u(x_0) = y_0, \end{cases} \tag{6}$$

in which u is to be determined as a solution of the differential equation $L[u] = 0$ subject to the initial condition $u(x_0) = y_0$ is well-posed: the solution given by (5) always exists; it depends continuously on y_0; and as we shall prove next, it is unique.

THEOREM 10-7. If p is continuous on an interval I containing the point x_0, then the initial value problem (6) has exactly one solution on I.

Proof. We have already verified that (6) has the solution given by (5). In order to prove uniqueness we employ the method of Theorem 8-5a. Let u be any solution of (6) within I on an interval containing x_0, and set

$$w(x) = \frac{u(x)}{v(x)} , \tag{7}$$

where

$$v(x) = \exp \left\{ - \int_{x_0}^{x} p(t)dt \right\} . \tag{8}$$

Since $v(x) > 0$, w is defined on the interval common to the domains of u and v . At the initial point x_0 , we have

$$w(x_0) = \frac{u(x_0)}{v(x_0)} = y_0$$

and we shall show now that w is a constant function, $w : x \longrightarrow y_0$ on its entire domain. Differentiating (7) we obtain (6) and (8)

$$\begin{aligned} w'(x) &= \frac{u'(x)v(x) - u(x)v'(x)}{v(x)^2} \\ &= \frac{[-p(x)u(x)]v(x) - u(x)[-p(x)v(x)]}{v(x)^2} \\ &= 0 . \end{aligned}$$

Thus w must be a constant function. Since $w(x_0) = y_0$, we conclude that $w : x \longrightarrow y_0$ on its domain. The function u is given by Formula (5) since $u = y_0 = v$.

The function v defined by (8) is called a fundamental solution of (4). (There is a fundamental solution for each choice of x_0 .) Any solution u of (4) is a constant multiple of v , namely $u = y_0 v$; consequently the family of solutions given by (5) is called the general solution of (4). Since the fundamental solution is everywhere positive we conclude that a solution u must be everywhere positive, or everywhere negative, or identically zero. The fundamental solution v clearly increases where $p(x) < 0$ and decreases where $p(x) > 0$; it is bounded if and only if $\int_{x_0}^{x} p(t)dt$ is bounded below; it approaches a constant state, that is, $\lim_{x\to\infty} v(x)$ exists, if as x approaches ∞, $\int_{x_0}^{x} p(t)dt$ either converges or approaches infinity. Since any solution u is proportional to the fundamental solution, the corresponding properties of u are immediately given

Example 10-7a. Consider the differential equation

$$y' - y \sin x = 0 .$$

Here $p(x) = -\sin x$, and for $x_0 = 0$

$$-\int_{x_0}^{x} p(t)dt = \int_{0}^{x} \sin t\, dt = 1 - \cos t .$$

The fundamental solution is then

$$v(x) = \exp(1 - \cos x) .$$

Clearly v is periodic with period 2π , increases on the interval $[0,\pi]$ and decreases on the interval $[\pi,2\pi]$. Its maximum value $v(\pi) = e^2$ is taken on at the points $x = (2k + 1)\pi$, its minimum $v(0) = 1$, at the points $x = 2k\pi$, where k is any integer.

(iii) The nonhomogeneous equation. The solution of the nonhomogeneous equation (3) is intimately related to the solution of the reduced equation. If u_1 and u are any solutions of the nonhomogeneous equation $L[u] = f$ then their difference $u - u_1$ is easily seen to be a solution of the homogeneous equation $L[u] = 0$ (Exercises 10-7, No. 2a). Consequently, given any solution z of (3), the general solution u can be written in the form

(9) $$u = z + cv$$

where c is a constant and v is a fundamental solution of the reduced Equation (4). Thus to obtain all solutions of (3) we have only to find one solution z , a particular solution, of $L[u] = f$, and a fundamental solution v of the reduced equation.

From Equation (9) we see again that the solutions of (3) form a one-parameter family; a single condition will serve to determine a value of the parameter c and a member of the family. Specifically, the initial value problem

(10) $$\begin{cases} L[u] = Du + p \cdot u = f \\ u(x_0) = y_0 \end{cases}$$

is solved by (9) with $c = \dfrac{y_0 - z(x_0)}{v(x_0)}$ for a particular solution z defined in some neighborhood of x_0 --provided such a solution exists. Furthermore, this solution is unique, for if z_1 and z_2 are solutions of (10) , then

$\zeta = z_2 - z_1$ is a solution of the initial value problem

$$L[\zeta] = 0 \ , \ \zeta(x_0) = 0 \ ,$$

and therefore ζ must be identically zero; hence, $z_1 = z_2$.

To show that the initial value problem (10) is well-posed we need only demonstrate the existence of a particular solution. For this purpose we apply an elegant device, the so-called method of variation of parameters, which applies to higher order linear equations as well.* The general solution of the reduced equation is $c\ v(x)$ where $v(x)$ is the fundamental solution. We seek a solution of the nonhomogeneous equation by "varying the parameter"; that is, we replace the constant c by a function c and seek a particular solution of the nonhomogeneous equation (3) in the form

$$z = c \cdot v \ . \tag{11}$$

Taking the derivative in (11) we obtain

$$z' = c' \cdot v + c \cdot v' \ .$$

In this relation we insert the conditions

$$z' = f - p \cdot z \quad , \quad v' = -p \cdot v$$

from Equations (3) and (4), respectively. This yields

$$c'(x) = \frac{f(x)}{v(x)} \ ,$$

whence

$$c(x) = \int_{x_0}^{x} \frac{f(t)}{v(t)}\, dt + k \ .$$

For each value of k this formula should define a particular solution of the form (11) and for simplicity we take $k = 0$. From (11) we then obtain as our candidate for a particular solution the function z given by

$$z(x) = c(x)v(x) = v(x) \int_{x_0}^{x} \frac{f(t)}{v(t)}\, dt \ . \tag{12}$$

It follows directly on differentiation that the function z defined by (12) is a solution of (3) on the common interval of continuity of f and p (Exercises 10-7, No. 4). From (9), the general solution u of the nonhomogeneous Equation (3) may be expressed in the form

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*The general method was invented by J.L. Lagrange (1736-1813), a French mathematician who contributed greatly to analysis and mechanics. The method for first order equations was given in 1697 by John Bernoulli (1667-1748), Swiss, one of the most important early developers of analysis.

(13) $$u(x) = v(x)[c + \int_{x_0}^{x} \frac{f(t)}{v(t)}\,dt] \;;$$

here c is constant. This is an explicit solution in the sense that it is expressed in terms of integrals of the given functions, but in general these integrals cannot be expected to have representations in terms of elementary functions.

Example 10-7b. Consider the differential equation for $y = u(x)$,

$$y' + ay = ke^{-bx} ,$$

where a, b and k are constants. In the notation of Equation (3), $p(x) = a$ and $f(x) = ke^{-bx}$. Take $x = 0$ in (8) to obtain for the fundamental solution

$$v(x) = \exp\left\{ \int_0^x (-a)dt \right\} = e^{-ax} .$$

In the general solution (13) set

$$I(x) = \int_0^x \frac{f(t)}{v(t)}\,dt = k \int_0^x e^{(a-b)t}\,dt .$$

We distinguish two cases. If $a = b$, we get $I(x) = kx$ and the general solution u is given by

$$y = u(x) = v(x)[c + I(x)] = (c + kx)e^{-ax} .$$

If $a \neq b$, then $I(x) = \frac{k}{a-b}[e^{(a-b)x} - 1]$ and here the general solution is

$$\begin{aligned} y &= e^{-ax}\left\{c + \frac{k}{a-b}[e^{(a-b)x} - 1]\right\} \\ &= Ae^{-ax} + \frac{k}{a-b}e^{-bx} \end{aligned}$$

where $A = c - \frac{k}{a-b}$ is constant.

For the particular solution of (3) which satisfies the initial condition $u(x_0) = y_0$ we obtain $u(x_0) = v(0)[c + 0] = c$ or $c = y_0$. Thus the explicit solution of the initial value problem (10) is

(14a) $$y = v(x)[y_0 + \int_{x_0}^{x} \frac{f(t)}{v(t)}\,dt]$$

where

$$v(x) = \exp\left\{-\int_{x_0}^{x} p(t)dt\right\} . \tag{14b}$$

Thus we have established that the initial value problem (10) is well-posed.

Example 10-7c. Consider a circuit consisting of a resistor and a coil subjected to an electromotive force, (see Figure 10-7). Let the resistance be R, the inductance of the coil be L and let the electromotive force be $E = g(t)$. We shall consider both constant and variable E. From Exercises 9-3, Number 8, Equation (5), the current $I = u(t)$ satisfies the differential equation

$$\frac{dI}{dt} + aI = f(t) \tag{15}$$

where

$$a = \frac{R}{L} \quad , \quad f(t) = \frac{g(t)}{L} = \frac{E}{L} .$$

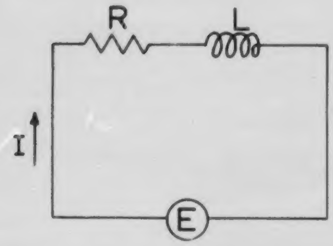


Figure 10-7

Equation (15) is a linear first order equation of the Form (3) with x replaced by t and y by I. For the general solution of (15) we have from (13)

$$I = e^{-at}[c + \int_0^t e^{as}f(s)ds] . \tag{16}$$

We now consider some of the problems which were left as exercises in Chapter 9.

(a) Consider the behavior of the circuit if the electromotive force is shunted out of the circuit at $t = 0$ when the current is I_0. Then for $t > 0$, $E = 0$ and I is given by the solution of the initial value problem for the homogeneous equation; namely,

$$I = I_0e^{-at} = I_0e^{-Rt/L} .$$

The current decays exponentially to 0. The half-life $\tau = \frac{L}{R}\log 2$, increases with L and decreases with R. In the limit of zero resistance the current is maintained constant at I_0. What happens in the limit of zero inductance, $L = 0$?

(b) Let the electromotive force E be constant from $t = 0$, on. If the initial current is I_0 then the circuit Equation (15) is non-homogeneous with a constant forcing term, $f(t) = \frac{E}{L}$. Then, by (14)

$$I = e^{-at}[I_0 + \frac{E}{L}\int_0^t e^{as}\, ds]$$

$$= \frac{E}{R} + (I_0 - \frac{E}{R})e^{-Rt/L} .$$

Thus the current approaches the constant terminal value $I_\infty = \frac{E}{R}$ as t approaches infinity. The system is said to approach the asymptotic steady state I_∞ and time-decaying term $I - I_\infty$ is called transient. The terminal value is independent of the initial state and is the value of the current that would flow in the circuit for $t > 0$ if no coil were present. Thus the effect of the coil is to level out the transition from the initial state to the asymptotic steady state.

(c) Let the initial current be I_0 and suppose for $t \geq 0$ there is an alternating electromotive force $E = E_0 \cos \omega t$. Set $b = \frac{E_0}{L}$ and $f(t) = b \cos \omega t$. From (14) we now have

$$I = e^{-at}[I_0 + b\int_0^t e^{as} \cos \omega s\, ds] .$$

Integration by parts, as in Example 10-4e, yields

$$I = \frac{b(a \cos \omega t + \omega \sin \omega t)}{a^2 + \omega^2} + e^{-at}(I_0 - \frac{a}{a^2 + \omega^2})$$

$$= \frac{E_0}{R^2 + \omega^2 L^2}(R \cos \omega t + \omega L \sin \omega t) + e^{-at}(I_0 - \frac{RL}{R^2 + \omega^2 L^2}) .$$

We may choose ϕ satisfying

$$\cos \phi = \frac{a}{a^2 + \omega^2} , \sin \phi = \frac{\omega}{a^2 + \omega^2} .$$

and write

$$I = \frac{E_0}{L}\cos(\omega t - \phi) + ke^{-at}$$

where k is constant (see Section A2-5). The asymptotic state of the circuit is a sinusoidal (alternating) current which has the same frequency as the forcing term but lags behind by an amount proportial to the phase ϕ.

Upon investigating the properties of the solution (14a, b) of the initial value problem we observe that the solution is continuously dependent on the initial datum y_0, as we might hope, but also that it is continuously dependent on the given functions p and f (Exercises 10-7, No. 9). This is an important observation for applications. The functions p and f may be empirical functions subject to the usual errors of measurement and interpolation. If these errors are kept small enough the error in the solution will be tolerable. We continually have drummed into our ears that a little knowledge is a dangerous thing, but in this instance we may take some consolation in the thought that a little ignorance need not be harmful.

Exercises 10-7

1. Verify the following properties of a linear operator L ,

$$L[au] = aL[u]$$
$$L[u + v] = L[u] + L[v]$$

for all functions u and v in the domain of L . Conversely, show if these properties are satisfied then L is a linear operator.

2. Let L be a linear operator.

   (a) Show that the difference between any two solutions of equation $L[u] = f$ is a solution of the homogeneous equation $L[u] = 0$. Show also that if u is a solution of $L[u] = f$ and v is a solution of the corresponding reduced equation, that $u + v$ is a solution of the original equation.

   (b) Verify that any linear combination of solutions of the homogeneous equation $L[u] = 0$, is again a solution.

   (c) Show if $c \neq 0$ that (2) cannot be put in the form of a linear equation

$$L[u] = f .$$

3. For the solution of Equation (4) in the Form (5) show that choice of a different end of integration $x_1 \neq x_0$ in the domain of p where $u(x_1) \neq 0$ yields the same value of y .

4. Verify that the function z defined by (12) is a particular solution of (3).

5. Give the general solutions of the following equations and solve the indicated initial value problem. In what domains are the solutions valid?

   (a) $y' + y = 4$; $y = 0$ at $x = 0$.

   (b) $y' = ay + b$, (a , b constants) ; $y = -\frac{b}{a}$ at $x = 0$, ($a \neq 0$) .

   (c) $xy' - y = x^2$; $y = 4$ at $x = 2$.

   (d) $2e^{-x}\,y' + e^{x}\,y = 2e^{x}$; $y = e + 2$ at $x = 0$.

6. (a) Obtain the general solution, in terms of elementary functions, of the equation

$$y' + xy = x .$$

   (b) Contrast with the general solution of

$$y' - xy = 1 .$$

7. Let u and v satisfy

$$u' + p \cdot u = f \ , \ v' + p \cdot v = g$$

where $f(x) > g(x)$ for $x > x_0$. Show that if $u(x_1) \geq v(x_1)$ for some $x_1 \geq x_0$, then $u(x) > v(x)$ for all $x > x_1$.

8. The solution of Equation (3) was obtained under the assumption that p and f are continuous functions. On the other hand, Formula (13) is meaningful when p and f may be only piecewise continuous (defined in Exercises 10-6b, No. 18). Revise the theory so that it applies more generally to piecewise continuous functions. (Hint: you will have to give up the requirement that solutions are differentiable at every point, but while relaxing this requirement, consider only solutions which are continuous.)

9. In the light of Number 6, determine the current in the circuit of Example 10-7c when the electromotive force is a "square wave" of period 2λ :

$$E(t) = \begin{cases} E_0 \ , \text{ for } \ 2n\lambda \leq t < (2n + 1)\lambda \ , \\ \\ -E_0 \ , \text{ for } \ (2n + 1)\lambda \leq t < (2n + 2)\lambda \ , \end{cases}$$

$n = 0 , 1 , 2 , \ldots$. What is the asymptotic solution as t approaches infinity?

10. Let $p(x) = \dfrac{a}{x^r}$ in the homogeneous Equation (4). Discuss the possibility of finding a solution $y = u(x)$ for $x > 0$ such that $\lim_{x \to 0^+} u(x)$ exists. What implications does this have for the initial value problem at $x = 0$?

Λ11. In the text it was stated that the solution depends continuously on the initial value y_0 and the functions p and f . The idea of continuous dependence on the initial values is clear: for a given value of x , y is a continuous (in this case, linear) function of y_0 . But what can be meant by continuous dependence on p and f ? Give your interpretation. Then verify that (14) does satisfy the continuous dependence conditions.

10-8. Linear Differential Equations of Second Order.

In this section we shall study differential equations of the general type

(1) $$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = f(x)$$

where p, q and f are all continuous on an interval I. Equations of this type appear in many areas of science and their theory is a large and still growing area of analysis. The Equation (1) involves the second derivative $\frac{d^2y}{dx^2}$ and no derivative of higher order; it is therefore an equation of second order. It is linear because it can be put in the form

(2a) $$L[u] = f$$

where $u(x) = y$ and L is the linear operator

(2b) $$L = D^2 + p \cdot D + q$$

(Exercises 10-8(a), No. 1).

Even if p, q and f are elementary functions, it is not always possible (in contrast to first order equations) to express the solutions of a linear equation of second order in terms of elementary functions of p, q, and f and their derivatives and integrals. Thus, a general discussion of these equations would lead us away from the theme of this chapter, technical integration and its uses. Here we shall discuss only the initial value problem for the general equation, and treat in detail only the special case, an extremely important one, for which p and q are constant functions (you will recall that in Section 8-5 we used an equation of this type, $y'' + y = 0$, to define the circular functions.)

We shall use the following simple properties of linear operators.

(a) If u_1 and u_2 are solutions of the homogeneous equation $L[u] = 0$, then any linear combination $c_1u_1 + c_2u_2$ is also a solution (Exercises 10-7, No. 2b).

(b) If u_1 and u_2 are solutions of the nonhomogeneous equation $L[u] = f$, i.e., $L[u_1] = f$ and $L[u_2] = f$, then $u_2 - u_1$ is a solution of the reduced equation; thus if we know one solution of the nonhomogeneous equation and all solutions of the reduced equation, we have all solutions of the nonhomogeneous equation (Exercises 10-7, No. 2a).

(c) If $f = c_1f_1 + c_2f_2$ and u_1 is a solution of $L[u] = f_1$ while u_2 is a solution of $L[u] = f_2$, then $c_1u_1 + c_2u_2$ is a solution of $L[u] = f$; this *superposition principle* often permits us to split a given problem into several simpler problems (Exercises 10-8a, No. 2).

(i) *Homogeneous equations with constant coefficients*. Consider the reduced equation of (1) with constant coefficients which we write in the form

$$L[u] = 0 \tag{3a}$$

where

$$L = D^2 + 2aD + b \, . \tag{3b}$$

It is natural to attempt to reduce the solution of Equation (3) to the solution of linear first order equations. This can be done if the operator L can be expressed as the "product" of first order operators. By the "product" LM of two differential operators we mean their composition: first apply M, then L; so $LM[u] = L[M[u]]$ (see Exercises 10-8a, No. 4). We seek constants α and β for which

$$L = (D - \alpha)(D - \beta) = D^2 + 2aD + b \, . \tag{4}$$

Once we succeed in this enterprise we can solve Equation (3) by solving in succession the equations

$$\begin{cases} (D - \alpha)v = 0 \\ (D - \beta)u = v \end{cases} \tag{5a}$$

for which we have developed general methods in the preceding section. (Compare Exercises 8-7, No. 14). Since the general solution of each first order equations involves an arbitrary constant, the solution of (5) will yield two arbitrary constants. This is true of second order equations in general. From the solution of (5) we expect a well-posed initial value problem to prescribe both the initial value $y_0 = u(0)$ and the derivative $y_0' = u'(0)$ (Exercises 10-8a, No. 5).

Observe that the composition of the operators $(D - \alpha)$ and $(D - \beta)$ behaves like ordinary multiplication; namely

$$\begin{aligned} (D - \alpha)(D - \beta)u &= (D - \alpha)[u' - \beta u] \\ &= D[u' - \beta u] - \alpha(u' - \beta u) \\ &= u'' - (\alpha + \beta)u' + \alpha\beta u \end{aligned}$$

or

$$(D - \alpha)(D - \beta) = D^2 - (\alpha + \beta)D + \alpha\beta \quad (5b)$$

In particular, the result is the same if α and β are interchanged so that linear differential operators with constant coefficients commute:

$$(D - \alpha)(D - \beta) = (D - \beta)(D - \alpha) .$$

The commutative property is usually not valid if α and β are not constants (Exercises 10-8a, No. 6). Comparing (5b) with (3b) we see that such a factorization is possible if the characteristic equation

$$\lambda^2 - 2a\lambda + b = 0 \quad (6)$$

has real roots, $\lambda = \alpha$ and $\lambda = \beta$. Consequently, for the purpose of solving Equation (3) we distinguish two cases: (1) the characteristic equation (6) has real roots, $a^2 - b \geq 0$: (2) the roots are complex, $a^2 - b < 0$.

We consider first the equation with a factored operator, namely

$$L[u] = (D - \alpha)(D - \beta)u = 0 . \quad (7)$$

We begin with the case $\alpha \neq \beta$; the equation for which the roots of the characteristic equation are equal will require separate treatment. Instead of solving (7) by means of the first order system (5), we shall employ a method which employs the commutativity of the first order factors of L . Observe that the equations $(D - \alpha)[v_1] = 0$ and $(D - \beta)[v_2] = 0$ have the solutions $v_1(x) = e^{\alpha x}$ and $v_2(x) = e^{\beta(x)}$ respectively. But clearly v_1 and v_2 are both solutions of (7) since

$$L[v_1] = (D - \beta)[(D - \alpha)v_1] = (D - \beta)[0] = 0$$

and

$$L[v_2] = (D - \alpha)[(D - \beta)v_2] = (D - \alpha)[0] = 0 .$$

Since any linear combination of solutions of the homogeneous equation is again a solution, we see that (7) has the two-parameter family of solutions

$$u(x) = c_1 e^{\alpha x} + c_2 e^{\beta x} . \quad (8)$$

We have not shown that (8) includes all solutions, but we know that we need two parameters to satisfy the two initial conditions suggested by (5). Furthermore, if the solution of the initial value problem is unique then no more than two parameters are needed. Formula (8) is in fact the general solution, but we offer no special treatment of the uniqueness question for the equation with constant coefficients because we shall prove uniqueness in general.

If the roots of the characteristic equation are equal, $\alpha = \beta = -a$, then the preceding method fails. The solution of the equation

$$(9) \qquad L[u] = (D + a)^2 u = 0$$

given by (8) is $u(x) = c_1 e^{-ax} + c_2 e^{-ax} = ke^{-ax}$, and we have essentially only one parameter, $k = c_1 + c_2$. (It is not always so easy to decide when a parameter is nonessential.) To find another solution of (9) we again use a technique from Section 10-7 and seek a solution in the form $u(x) = e^{-ax}v(x)$ (see Miscellaneous Exercises, Chapter 10, No. 23). When the operator $D - a$ is applied to this function, we get the simplifying result,

$$(D + a)[e^{-ax}v(x)] = e^{-ax}D\,v(x) .$$

Now, apply the operator again to obtain

$$\begin{aligned} L[u(x)] &= (D + a)^2[u(x)] \\ &= (D + a)^2[e^{-ax}v(x)] \\ &= (D + a)[e^{-ax}D\,v(x)] \\ &= e^{-ax}D^2v(x) . \end{aligned}$$

Thus, if u is a solution of (9), then v satisfies the differential equation

$$e^{-ax}D^2v = 0 .$$

Since e^{-ax} is always positive, we have $D^2v = 0$. Now we integrate twice to obtain $v(x) = c_1 + c_2x$ where c_1 and c_2 are arbitrary constants. Thus we obtain the general solution of (9)

$$(10) \qquad u(x) = e^{-ax}(c_1 + c_2x) .$$

Now we turn our attention to the solution of (3) when the roots of the characteristic equation (6) are complex, i.e., when $a^2 - b < 0$. In that case we complete the square to obtain

$$L = D^2 + 2aD + b = (D + a)^2 + b - a^2 .$$

Since $b - a^2$ is positive we may set $b - a^2 = \omega^2$ and write the differential equation in the form

$$(11) \qquad L[u] = \Big((D + a)^2 + \omega^2\Big)u = 0 .$$

In the preceding case we saw that the substitution $u(x) = e^{-ax}v(x)$ is useful in relation to the operator $(D + a)^2$. We use the same method here and get the differential equation for v

$$e^{-ax}(D^2 + {}^2 v) = 0$$

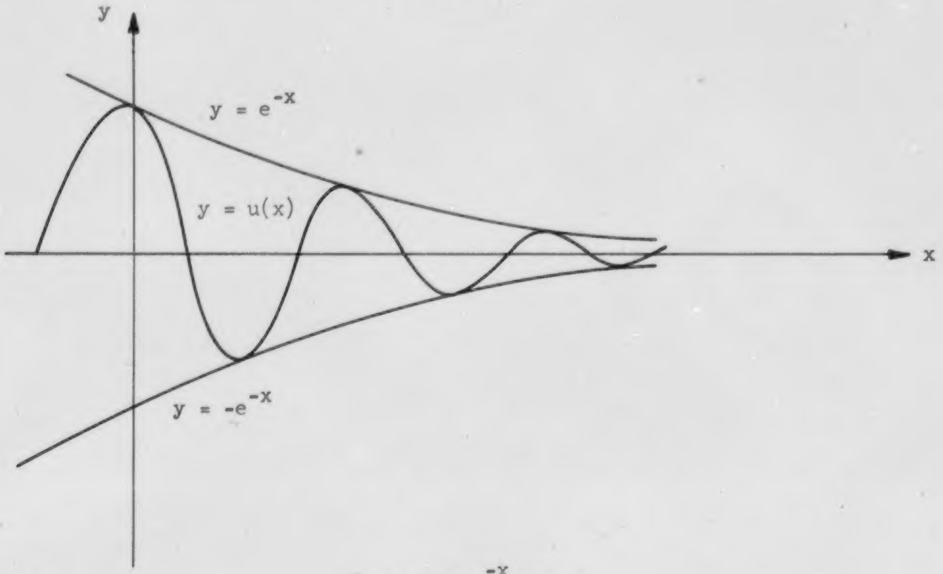
where we may factor out e^{-ax} since it is positive. The equation for v is clearly kin to the equation $y'' + y = 0$ for the circular functions which we investigated in Section 8-5; it has the solutions $\sin \omega x$ and $\cos \omega x$ (Exercises 8-5, No. 15). Thus (11) has the solutions $e^{-ax} \sin \omega x$ and $e^{ax} \cos \omega x$, hence any linear combination of these is a solution:

(12a) $$u(x) = e^{-ax}(c_1 \cos \omega x + c_2 \sin \omega x) .$$

It is often convenient to introduce the parameters $A = \sqrt{c_1{}^2 + c_2{}^2}$ and ϕ defined by $\cos \phi = \frac{c_1}{A}$, $\sin \phi = \frac{c_2}{A}$; in terms of these parameters the solutio has the form

(12b) $$u(x) = Ae^{-ax} \cos (\omega x - \phi) .$$

If $a > 0$, the graph of this solution is an exponentially damped sinusoid (Figure 10-8); this case is important in many applications and we shall make use of it in Chapter 12 (see Exercises 10-8a, No. 9).



$f : x \longrightarrow e^{-x} \cos x$

Figure 10-8

In obtaining the general solutions of all linear differential equations of second order we have used a number of tricks, albeit more than once, whose motivation must seem obscure. There is an old saw among mathematicians that a device used once is a trick; used twice, it becomes a technique; three times a method; more than that, it becomes a rule and demands a theory. In fact, the devices employed here all have more general uses in the further development of the theory of linear differential equations.

Exercises 10-8a

1. Verify that the operator

$$L = D^2 + p \cdot D + q$$

is linear.

2. Prove the superposition principle for linear operators.

3. Prove that the reduced equation $L[u] = 0$, L linear, always has the trivial solution $u(x) = 0$.

4. If L and M are linear operators, show that the composition LM is linear.

5. Take $L = (D - \alpha)(D - \beta)$.

   (a) Solve the homogeneous equation $L[u] = 0$ by the method indicated in (5).

   (b) Show that the initial value problem $L[u] = 0$ with the prescribed initial data, $u(x_0) = y_0$, $u'(x_0) = y_0'$, is well-posed.

6. Show that the differential operators $D - p$ and $D - q$ commute if and only if p and q differ by a constant.

7. Find the general solutions of the following differential equations and obtain solutions of the stated initial value problems.

   (a) $D^2u = 0$; $u(0) = 1776$, $u'(0) = 1929$.

   (b) $(D^2 + 2D + 2)u = 0$; $u(0) = \frac{\sqrt{2}}{2}$, $u'(0) = 0$.

   (c) $(D^2 + 2D - 2)u = 0$; $u(0) = 0$, $u'(0) = \sqrt{3}$.

   (d) $(D^2 + D - 2)u = 0$; $u(0) = 1$, $u'(0) = 2$.

8. In analogy with (12) express the solution of (7) in terms of hyperbolic functions.

9. Discuss the graph of the damped sinusoidal function u of (12) for $a = 1$, $\omega = 1$, $u(0) = 1$, $u'(0) = 1$. Pay particular attention to zeros, extrema, points of inflection and points where the graph touches the curves $y = \pm e^{-x}$.

10. (a) Show if the roots of the characteristic equation (6) are complex than all solutions of the homogeneous equation (3a) have graphs obtained from some nontrivial solution by a charge of scale along the y-axis, $(x,y) \longrightarrow (x,Cy)$, and a translation in the direction of the x-axis, $(x,y) \longrightarrow (x + \xi , y)$.

(b) Investigate the other two classes of solutions of (3a) with respect to this geometrical property.

(ii) The general homogeneous equation. The question of uniqueness for the homogeneous second order linear equation with constant coefficients was left unresolved. In this section, we shall prove uniqueness of the solution of the initial value problem for the reduced equation of (1),

$$L[u] = D^2u + p \cdot Du + q \cdot u = 0 \tag{13a}$$

with p and q continuous, subject to the initial conditions

$$u(x_0) = y_0 \quad , \quad u'(x_0) = y_0' \ . \tag{13b}$$

Existence of a solution is another question. For the equation with constant coefficients existence was settled simply by exhibiting an appropriate general solution. For Equation (13) it is known that there is generally no explicit solution of the types available to us by methods of this course. It is possible to prove that such a solution exists, nonetheless, without knowledge of an explicit representation. In the general theory of differential equations existence is demonstrated by the construction of a solution as the limit of a convergent approximation scheme. Here we assume without proof that a solution of the initial value problem exists and simply investigate the structure of the solution.

Actually, we only assume that on the interval I of continuity of p and q Equation (13a) has solutions ϕ and ψ which satisfy the specific initial conditions

$$\phi(x_0) = 1 \quad , \quad \phi'(x_0) = 0 \ , \tag{14a}$$

$$\psi(x_0) = 0 \quad , \quad \psi'(x_0) = 1 \ , \tag{14b}$$

for just one point x_0 in I. (Actually, it is sufficient to assume that (13a) has any nontrivial solution on I (see Miscellaneous Exercises, Chapter 10., Nos. 23, 26). A pair of functions $\{\phi, \psi\}$ satisfying conditions (14a,b) is called a fundamental set of solutions of the reduced equation. If two such solutions exist then the initial conditions are satisfied by the linear combination

$$u = y_0\phi + y_0'\psi \ .$$

If uniqueness is proved then each solution is a linear combination of ϕ and ψ.

Now let us suppose that the initial conditions

$$u(x_1) = y_1 \quad , \quad u'(x_1) = y_1' \tag{15}$$

are prescribed at some other point x_1 of I . Since $u = c_1\phi + c_2\psi$ is a solution of (14a) for any constants c_1 and c_2 we try to satisfy the initial conditions (15) by picking suitable values of c_1 and c_2 . Therefore, we require

$$\begin{cases} c_1\phi(x_1) + c_2\psi(x_1) = y_1 \\ c_1\phi'(x_1) + c_2\psi'(x_1) = y_1' \, . \end{cases} \tag{16}$$

Equation (16) can be solved for c_1 and c_2 if the determinant

$$\phi(x_1)\psi'(x_1) - \phi'(x_1)\psi(x_1)$$

is not zero. Since we are interested in solving (16) for any value x_1 in I we are led to consider the function w given by

$$w(x) = \phi(x)\psi'(x) - \phi'(x)\psi(x) \tag{17}$$

and to inquire whether w ever takes the value zero.*

From the conditions of (14a,b) we have $w(x_0) = 1$. Since w is continuous it is positive on some nieghborhood of x_0 . We can do much better. Surprisingly, even though we know nothing much about the solutions ϕ and ψ beyond their bare existence, we can write a simple explicit formula for w . For this purpose we compute w' , using the differential equation (13a) to express ϕ'' and ψ'' in terms of lower order derivatives:

$$\begin{aligned} w' &= (\phi' \cdot \psi' + \phi \cdot \psi'') - (\phi'' \cdot \psi + \phi' \cdot \psi') \\ &= \phi \cdot \psi'' - \psi \cdot \phi'' \\ &= \phi \cdot (-p \cdot \psi' - q \cdot \psi) - \psi \cdot (-p \cdot \phi' - q \cdot \phi) \\ &= -p \cdot (\phi \cdot \psi' - \phi' \cdot \psi) \\ &= -p \cdot w \, . \end{aligned}$$

Thus w satisfies the linear homogeneous equation of first order

$$Dw + p(x)w = 0 \tag{18a}$$

and the initial condition

$$w(x_0) = 1 \, . \tag{18b}$$

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[*] The function w is usually called the Wronskian of ϕ and ψ after H. Wronski (c. 1821).

From Section 10-7, we have immediately,

$$w(x) = \exp\left\{-\int_{x_0}^{x} p(t)dt\right\} . \tag{18c}$$

The explicit expression of w is less important to us than the knowledge derived from the exponential representation that $w(x) > 0$ for all x. We conclude that constants c_1 and c_2 satisfying (16) can be found and, hence, that the initial value problem (13a,b) has a solution for any point of I and any choice of initial values.

THEOREM 10-8. The initial value problem (13) has at most one solution.

Proof. Let u_1 and u_2 be distinct solutions of the initial value problem (13). Since $u_2(x) - u_1(x)$ is not identically zero, it follows that $u = u_2 - u_1$ is a nontrivial solution of the initial value problem

$$L[u] = 0 \ ; \ u(x_0) = 0 \ , \ u'(x_0) = 0 \ . \tag{19}$$

Now, if

$$w_1(x) = \phi(x)u'(x) - u(x)\phi'(x),$$

where ϕ is the solution given in (14a), then by the same derivation as that of (18c) from (17)

$$w_1(x) = w_1(x_0) \exp\left\{-\int_{x_0}^{x} p(x)dx\right\} .$$

From the initial conditions on u given in (19), we have $w_1(x_0) = 0$ and hence that $w_1(x)$ vanishes identically; consequently,

$$\phi(x)u'(x) = u(x)\phi'(x) \ . \tag{20a}$$

By exactly the same argument for the function ψ of (14b),

$$w_2(x) = \psi(x)u'(x) - u(x)\psi'(x)$$

vanishes identically; hence

$$u(x)\psi'(x) = \psi(x)u'(x) \ . \tag{20b}$$

Multiplying corresponding sides of (20a) and (20b) we obtain

$$[u(x)u'(x)][\phi(x)\psi'(x)] = [u(x)u'(x)][\psi(x)\phi'(x)]$$

whence,

$$[u(x)u'(x)][\phi(x)\psi'(x) - \psi(x)\phi'(x)] = \frac{1}{2} w(x) \, D[u(x)^2] = 0 .$$

Since $w(x) > 0$ for all x, it follows that $D[u(x)^2] = 0$, or that $u(x)^2$ is constant. Since $u(x_0) = 0$, we conclude that $u(x) = 0$ for all x, in contradiction to the assumption that u was a nontrivial solution of (19).

Corollary. Every solution of $L[u] = 0$ in I is a linear combination of the functions ϕ and ψ defined by the initial conditions (14).

Proof. Let ξ be any point of the domain of u and set $u(\xi) = y_0$, $u'(\xi) = y_0'$. By the Theorem 10-8 we know that u is the only solution with these initial values. At the same time we have established that there exists a linear combination $c_1\phi + c_2\psi$ which has the same initial values. Thus $u = c_1\phi + c_2\psi$.

Under the assumption of the existence of the two solutions ϕ and ψ which satisfy the initial conditions (14a,b) at any one point x_0 of the interval I where p and q are continuous, we have proved that the initial value problem (13) for the general homogeneous second order linear equation is well posed: a solution of (13a) satisfying the initial conditions (13b) exists and is uniquely defined. From the preceding analysis it also follows that the solution is a linear function of the initial data; hence continuous dependence on the initial data is immediate.

(iii) The general nonhomogeneous equation. As for the first order linear equation, the solution of the general nonhomogeneous second order linear equation (1) can be expressed entirely in terms of the reduced equation.

First we show that the initial value problem has at most one solution. Let $y_1 = u_1(x)$ and $y_2 = u_2(x)$ be two solutions of the initial value problem

$$L[u] = f \; ; \; u(x_0) = y_0 \, , \; u'(x_0) = y_0' , \tag{21}$$

where L is the general linear second order operator (2b). Then $v(x) = u_2(x) - u_1(x)$ is a solution of the initial value problem (19) for the homogeneous equation with zero initial data. But we have shown that $v(x)$

must then be identically zero. It follows that $u_1 = u_2$, hence that the solution of the initial value problem is unique if any solution exists.

Next, we show that if (1) has any solution then a solution of the initial value problem (21) exists. Suppose $L[z] = f$ where $z(x_0) = \alpha$, $z'(x_0) = \beta$. There exists a solution v of the homogeneous equation $L[v] = 0$ for which $v(x_0) = y_0 - \alpha$ and $v'(x_0) = y_0' - \beta$. Consequently, for $u = v + z$ by the superposition principle, $L[u] = L[v + z] = f$; furthermore, $u(x_0) = v(x_0) + z(x_0) = y_0$ and $u'(x_0) = v'(x_0) + z'(x_0) = y_0'$, so that u is a solution of the initial value problem (21). We already know it is the only possible one.

Now, somehow, we must find a single particular solution of Equation (1). We know that if $\{\phi, \psi\}$ is a fundamental set of solutions for the reduced equation $L[u] = 0$, then any solution has the form $c_1\phi + c_2\psi$. For convenience in solving the initial value problem (21) we choose $\{\phi, \psi\}$ as the particular fundamental set satisfying conditions (14a,b) at the point x_0 where the initial data are prescribed. Again we try Lagrange's rule of variation of parameters* and seek a particular solution in the form

$$v(x) = c_1(x)\phi(x) + c_2(x)\psi(x) . \tag{22}$$

We require $L[u] = f$. Calculating $L[u]$ from (22) and using $L[\phi] = L[\psi]$ we obtain

$$L[u] = (c_1'' \cdot \phi + c_2'' \cdot \psi) + 2(c_1' \cdot \phi' + c_2' \cdot \psi') + p \cdot (c_1' \cdot \phi + c_2' \cdot \psi) . \tag{23}$$

Since we have two functions, c_1 and c_2, to be determined we may impose two conditions. We already have imposed one, $L[u] = f$ and we are free to impose another so as to simplify (23) insofar as we can. Observe that the derivative of the third parenthetic expression in (23) is the sum of the other two:

$$D(c_1' \cdot \phi + c_2' \cdot \psi) = (c_1'' \cdot \phi + c_2'' \cdot \psi) + (c_1' \cdot \phi' + c_2' \cdot \psi) .$$

We impose the condition that $c' \cdot \phi + c_2' \cdot \psi = 0$; then the derivative vanishes also, and $L[u] = c_1' \cdot \phi' + c_2' \cdot \psi'$. Thus, we impose the two conditions on c_1 and c_2,

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*We see now that Lagrange's method has become a rule and demands a theory. The theory, a relatively recent development, requires somewhat more subtle analytical techniques than we employ here. It is based on the beautiful theory of linear vector spaces (usually found under the headings of "matrices" or "linear algebra").

$$\begin{cases} c_1' \cdot \phi + c_2' \cdot \psi = 0 \\ c_1' \cdot \phi' + c_2' \cdot \psi' = f \; . \end{cases}$$

So long as the determinant $w = \phi \cdot \psi' - \phi' \cdot \psi$ is not zero, this system can be solved for c_1' and c_2' (compare the system (16)); namely,

$$c_1'(x) = -\frac{\psi(x)}{w(x)} f(x) \quad , \quad c_2'(x) = \frac{\phi(x)}{w(x)} f(x) \; .$$

We integrate these expressions from x_0 to x to obtain $c_1(x)$ and $c_2(x)$ and obtain a particular solution in the form (22):

$$v(x) = -\phi(x) \int_{x_0}^{x} \frac{\psi(t)}{w(t)} f(t)dt + \psi(x) \int_{x_0}^{x} \frac{\phi(t)}{w(t)} f(t)dt \; . \tag{24}$$

As an exercise, prove this result and show that (24) gives a particular solution by differentiating and verifying directly that $L[v] = f$ (Exercises 10-8b, No. 3). Furthermore, this solution satisfies the initial conditions

$$v(x_0) = 0 \; , \; v'(x_0) = 0 \; , \tag{25}$$

as you may easily check. Had we not chosen both lower ends of integration to be x_0 (as we are free not to do), we would generally not obtain initial conditions so convenient as (25) for the solution of the initial value problem (21) (Exercises 10-8b, No. 4). As it is, we obtain the solution of the initial value problem at once in the form

$$u(x) = y_0\phi(x) + y_0'\psi(x) + v(x) \; . \tag{26}$$

Again it is obvious from the linear dependence of $u(x)$ on the initial data that the solution satisfies the property of continuous dependence on the initial values. Furthermore, continuous dependence on the forcing term f is also apparent (see Exercises 10-7, No. 11). However, to prove continuous dependence on p and q requires the existence theory which we are assuming.

An interesting form of (24) is obtained by combining the two integrals:

$$v(x) = \int_0^x \frac{\psi(x)\phi(\xi) - \phi(x)\psi(\xi)}{w(\xi)} f(\xi)d\xi \; .$$

The first factor appearing in the integrand,

$$G(x,\xi) = \frac{\psi(x)\phi(\xi) - \phi(x)\psi(\xi)}{w(\xi)} \; , \tag{27}$$

is called the Green's function of L ; it has some striking properties:

(a) For any fixed number ξ, the function $u : x \longrightarrow G(x,\xi)$ is a linear combination of $\phi(x)$ and $\psi(x)$; therefore $G(x,\xi)$ is a solution of the reduced equation $L[u] = 0$.

(b) $G(\xi,\xi) = 0$.

For the derivative $u'(\xi) = D_x G(x,\xi)\Big|_{x=\xi}$, we obtain

(c) $D_x G(x,\xi)\Big|_{x=\xi} = \dfrac{\psi'(\xi)\phi(\xi) - \phi(\xi)\psi'(\xi)}{W(\xi)} = 1$.

In summary, $G(x,\xi) = u(x)$ is that solution of the reduced equation which satisfies the initial conditions $u(\xi) = 0$ and $u'(\xi) = 1$. It follows that $G(x,\xi)$ is uniquely determined and does not depend on the process used to construct it.

The remarkable property of $G(x,\xi)$ which we have just derived has great value, since $G(x,\xi)$ is the fixed object in the integral which defines the particular solution v; $G(x,\xi)$ does not depend on the forcing term f or the lower end of integration x_0.

The integral

(28a)
$$v(x) = \int_{x_0}^{x} G(x,\xi)f(\xi)d\xi$$

defines an integral operator

(28b)
$$T : f \longrightarrow v$$

which has the effect of transforming any forcing function into that particular solution of $L[u] = f$ which has zero initial values at $x = x_0$. The relation of the integral operator T to the differential operator L generalizes the relation between the operation of integration and the operation of differentiation: if S denotes the operator for ordinary integration; that is, $S[f] = F$ where $F(x) = \int_{x_0}^{x} f(\xi)d\xi$, then for a continuous function f,

(29a)
$$DS[f] = D[F] = f ;$$

and for any continuously differentiable function F,

(29b)
$$SD[F(x)] = F(x) - F(x_0) .$$

From (29a,b) we see that S and D are almost, but not exactly, inverse operators. From (29a) we see that D undoes the work of S, but from (29b) we see that S does not quite invert the effect of D, but adds a constant function. (If the domain of D is restricted to functions satisfying $F(x_0) = 0$ then the operators become exact inverses). Similarly, for any continuous function f,

$$LT[f] = L[v] = f, \tag{30}$$

and for any twice continuously differentiable function u,

$$TL[u] = u - y_0\phi - y_0'\psi, \tag{30b}$$

where $y_0 = u(x_0)$ and $y_0' = u'(x_0)$. (Again the operators are exact inverses if the domain of L is restricted to functions having zero initial data at x_0).

In terms of the symbolic operator description, the problem $L[u] = f$ is solved for a suitable restriction of the domain of L by finding an inverse operator T such that $TL[u] = TF = u$ (see Exercises 10-8b, No. 5). This symbolic formulation describes a general class of problems which play a central role in higher mathematics. For linear operators there is an elegant well-rounded theory contained in the areas of linear algebra and linear analysis, but even these areas have not been fully explored and are still growing robustly. The representation of inverse operators in terms of Green's functions (or analogous forms) is a useful method in much of this theory.

Exercises 10-8b

1. Show for each of the special cases (7), (8), (9) of Equation (3) that there exist solutions ϕ and ψ satisfying the initial conditions

$$\begin{cases} \phi(0) = 1 \\ \phi'(0) = 0 \end{cases} \qquad \begin{cases} \psi(0) = 0 \\ \psi'(0) = 1 , \end{cases}$$

hence that the initial value problem is well-posed.

2. Find a fundamental set of solutions of

$$L[u(x)] = u''(x) + xu'(x) = 0$$

and solve the initial value problems

(a) $L[u(x)] = 1 , \ u(0) = u'(0) = 0 .$

(b) $L[u(x)] = x , \ u(0) = u'(0) = 0 .$

3. Verify directly by differentiation that (24) defines a particular solution of (1) and show that the zero initial conditions (25) are satisfied.

4. Show how to solve the initial value problem (21) when the lower ends of integration in the expression (25) for the particular solution may be different from x_0 .

5. Describe the solution of the initial value problem for the general second-order linear equation (1) in operator symbolism by suitably restricting the domain of the differential operator and giving the exact inverse integral operator.

6. Determine the Green's function for the operator with constant coefficients.

7. The theory of Equation (13a) is based on the assumption that p and q are continuous. This theory can be extended to piecewise continuous functions p and q , if the requirement that solutions be twice continuously differentiable is weakened to require that they be only once continuously differentiable. Assuming the validity of this assertion, construct the fundamental set of solutions at $x = 0$ for the following equations:

(a) $y'' + (\operatorname{sgn} x)y = 0 .$

(b) $y'' + k(x)y = 0$, where

$$k(x) = \begin{cases} 1 , \text{ if } & 2n\pi \leq x < (2n + 1)\pi \\ 5 , \text{ if } & (2n + 1)\pi \leq x < (2n + 2)\pi , \end{cases}$$

$$(n = 0, \pm 1 , \pm 2 , \ldots) .$$

8. (a) Construct the asymptotic solution as x approaches infinity for the equation

$$y'' + 4y' + 3 = k(x)$$

where k is defined in Number 6.

(b) Do the same for the equation

$$y'' + 2y' + 2 = k(x) .$$

10-9. Separable Differential Equations.

In this section we treat another broad class of differential equations which can be solved in terms of integrals, equations of the form

$$\frac{dy}{dx} = f(x)g(y) \tag{1}$$

where g and f are given continuous functions. This class includes the equation $y' = f(x)$ whose study is the principal objective of this chapter; the homogeneous linear equation of first order, $y' = -p(x)y$, which we treated in Section 10-7; and the equation $y' = a + by + cy^2$ which served as the principal mathematical model for the processes of growth, decay, and competition considered in Chapter 9.

Equation (1) contains only a first derivative and therefore is of order 1. We may then expect on the ground of our experience with first-order equations that it is appropriate to pose the initial value problem: to determine that solution of (1) for which $y = y_0$ when $x = x_0$. Differential equations of this form are generally nonlinear (see Exercises 10-7, No. 2c). Since they are easy to handle, we shall use them as a means of gaining some insight into some of the questions which arise in connection with nonlinear equations.

We solve (1) by a formal procedure and call attention along the way to the difficulties which may arise. If $y = u(x)$ is a solution of (1), then

$$u'(x) = f(x)g\big(u(x)\big) .$$

We may divide by $g\big(u(x)\big)$, provided that $g\big(u(x)\big) \neq 0$, to obtain

$$\frac{u'(x)}{g(u(x))} = f(x) . \tag{2}$$

Now, upon integrating with respect to x, we have

$$\int \frac{u'(x)}{g(u(x))}\, dx = \int f(x)dx$$

or, in Leibnizian notation,

$$\int \frac{dy}{g(y)} = \int f(x)dx . \tag{3}$$

If G is any antiderivative of $\frac{1}{g}$, and F any antiderivative of f, then (3) is equivalent to

$$G(y) = G\big(u(x)\big) = F(x) + C . \tag{4}$$

This equation does not give y explicitly. In order to determine y we must find the inverse H of G if it exists; then we shall have

$$y = H\big(F(x) + C\big) . \tag{5}$$

Then, so long as $F(x) + C$ is in the domain of H, Equation (5) defines a solution of (1). If the value of u is prescribed at x_0,

$$u(x_0) = y_0 , \tag{6a}$$

then, by (4), $C = G(y_0) - F(x_0)$, and from (5) we obtain as the solution of this initial value problem

$$y = H\big(F(x) - F(x_0) + G(y_0)\big) . \tag{6b}$$

The method of obtaining (3) is called *separation of variables* and, accordingly, Equation (1) is called *separable*.

Example 10-9a. Given a one-parameter family of curves, its *orthogonal trajectories* are defined as curves which cross the members of the family only at right angles. Thus the straight lines through the origin are orthogonal trajectories to the circles $x^2 + y^2 = a^2$.

Let us consider the problem of finding the orthogonal trajectories to the family of parabolas

$$y = ax^2 .$$

Observe first, that for each point (ξ,η) of the plane except for the points of the y-axis, there is exactly one member of the family passing through the point, the parabola given by $a = \frac{\eta}{\xi^2}$. At this point, the parabola has the slope $2a\xi = \frac{2\eta}{\xi}$. Thus an orthogonal trajectory passing through (ξ,η) must have the slope $\frac{-\xi}{2\eta}$. If $y = u(x)$ is an orthogonal trajectory to the family of parabolas, we conclude that at each point of the orthogonal trajectory

$$y' = -\frac{x}{2y} . \tag{7}$$

Thus the orthogonal trajectories satisfy a separable differential equation. Separating variables, we have,

$$2yy' = -x ,$$

and integrating, we obtain

$$y^2 = -\frac{x^2}{2} + C \tag{8a}$$

or

$$\text{(8b)} \qquad \frac{x^2}{2} + y^2 = C \ .$$

This formula describes the family of ellipses centered at the origin with ratio of major axis to minor axis equal to $\sqrt{2}$.

On comparing (8a) with (4) we observe that the function $G : y \longrightarrow y^2$ does not have an inverse on the domain of real numbers. On any interval where $y \neq 0$, however, G does have an inverse. When $y = 0$, Equation (7) becomes singular, although (8b) remains geometrically significant. Finally, note that not all values of the parameter C are admissible; only $C > 0$ yields a solution.

Example 10-9b. Consider the equation

$$y' = xy \ .$$

If $y \neq 0$ (but, $y = 0$ is definitely a solution), then

$$\frac{y'}{y} = x \ ,$$

whence

$$\log |y| = \frac{x^2}{2} + C \ .$$

Here, too, the function $G : y \longrightarrow \log |y|$ does not have an inverse on its domain of all nonzero reals. On any interval which does not contain $y = 0$ we obtain the solution

$$y = \exp\left\{\frac{x^2}{2} + C\right\} \operatorname{sgn} y \ ,$$

where sgn y is constant under the restriction on y . We have lost the solution $y = 0$, but if we set $k = e^C \operatorname{sgn} y$, we obtain all solutions in the form $y = ke^{x^2/2}$. The method of separation of variables, however, does not by itself necessarily yield all solutions.

Aside from the special cases arising from possible zeros and points of discontinuity of $g(y)$, the theory of separable equations is extremely simple.

THEOREM 10-9. Let f be continuous on a neighborhood of x_0, and g continuous on a neighborhood of y_0. If $g(y_0) \neq 0$, then the initial value problem for $y = u(x)$,

$$\begin{cases} \frac{dy}{dx} = f(x)g(y) \\ u(x_0) = y_0 \end{cases}$$

has exactly one solution.

Proof. Since $g(y_0) \neq 0$, we know by Lemma 3-4 that $g(y)$ is bounded away from zero on some neighborhood I of y_0. We conclude that $\frac{1}{g(y)}$ is continuous and has constant sign on I. The function G given by

(9a)
$$G(y) = \int_{y_0}^{y} \frac{1}{g(s)}\, ds$$

is strongly monotone on I, because $G'(y) = \frac{1}{g(y)} \neq 0$. Observe also that

(9b)
$$F(x) = \int_{x_0}^{x} f(t)dt$$

is continuous. Since $F(x_0) = G(y_0) = 0$ it follows that there exists a neighborhood of J of x_0 which is mapped by F into the range of G over I, that is, into the domain of the inverse function H. We conclude that the function

(9c)
$$u : x \longrightarrow H\big(F(x)\big)$$

is defined on J. From the differentiation theorems for compositions and inverses

$$u'(x) = \frac{F'(x)}{G'(y)} = f(x)g(y) ;$$

so that u is a solution of the differential equation. Furthermore,

$$u(x_0) = H\big(F(x_0)\big) = H(0) = y_0 ,$$

since $G(y_0) = 0$. Thus the existence of a solution of the initial value problem is proved.

On the other hand, if a solution of the initial value problem exists, then the method of separation of variables is justified and the solution is dictated by Formulas (9a,b,c). Thus the solution is unique.

It is possible to relax the condition that $g(y_0) \neq 0$ in Theorem 10-9, but then we must impose a stronger condition on g, for example, that g have a bounded derivative rather than be merely continuous. We assume without proof that this condition holds and ask what form the solution may have when $g(y_0) = 0$. The constant function $u : y \longrightarrow y_0$ is then clearly a solution of the differential equation, and, in view of the assumed uniqueness under the restriction on g, it is the only one.

Exercises 10-9

1. Solve the equation

$$y' + xy = x$$

by separating variables. (Compare Exercises 10-7, No. 6).

2. Find the orthogonal trajectories to each of the following families of curves and sketch the curves and their orthogonal trajectories.
   (a) The rectangular hyperbolas, $xy = a$.
   (b) The ellipses centered at the origin with fixed ratio of major to minor axis.

3. (a) Show that the equation

$$\frac{dy}{dx} = \phi(\frac{y}{x})$$

is transformed into a separable equation in $\frac{dy}{dv}$ by means of the substitution $x = \frac{y}{v}$.

   (b) Find the orthogonal trajectories to the family of circles $(x - a)^2 + y^2 = a^2$.

Miscellaneous Exercises

1. Integrate.

$$\int a^x b^x dx \ .$$

2. Integrate.

$$\int \frac{e^{2x}}{\sqrt[4]{e^x + a^2}}\, dx \ .$$

3. Evaluate

$$I = \int_0^\infty \frac{x\, dx}{x^4 + x^2 + 1} \ .$$

4. Evaluate

$$\int_0^a \frac{dx}{x + \sqrt{a^2 - x^2}} \ .$$

5. Prove that

$$\int_a^\infty \frac{dx}{x^4\sqrt{a^2 + x^2}} = \frac{2 - \sqrt{2}}{3a^4} \ .$$

6. Compare $\int_0^1 \frac{dx}{x^{1-\epsilon}}$ with $\int_1^\infty \frac{dx}{x^{1+\epsilon}}$.

7. Evaluate $\int_{-1}^1 \sqrt{\frac{1 + x}{1 - x}}$.

8. Show that

$$2\int_0^\infty G(t + \sqrt{t^2 + 1})dt = \int_1^\infty (1 + \frac{1}{t^2})G(t)dt \ ,$$

assuming that the integral exists.

9. Prove that

$$\int_0^\infty (\sqrt{t^2 + 1} - t)^n dt = \frac{n}{n^2 - 1} \quad \text{provided} \quad n > 1 \ .$$

(Hint: Use No. 8.)

10. Compute

$$I = \int_0^\infty \frac{dx}{(x^2 + a^2)(x^2 + b^2)} , \; a > 0 , \; b > 0 , \; a \neq b .$$

11. Compute

$$\int_0^\infty \frac{x\,dx}{(x^2 + a^2)(x^2 + b^2)} , \; a > 0 , \; b > 0 .$$

12. Show that the integrals

(a) $$\int_e^\infty \frac{dx}{x(\log x)^\alpha}$$

(b) $$\int_{e^e}^\infty \frac{dx}{x(\log x)(\log \log x)^\alpha}$$

converge if $\alpha > 1$ and diverge if $\alpha \leq 1$, and evaluate the integrals.

13. Prove that $\int_0^1 \sin \frac{1}{x}\,dx$ converges.

14. (a) Prove that if f has a continuous derivative then

$$\int_1^x [\![t]\!] f'(t)dt = [\![x]\!] f(x) - \sum_{n=1}^{[\![x]\!]} f(n) \qquad ([\![t]\!] = \text{integral part of } t) .$$

(b) Evaluate $\int_1^x [\![t]\!] t^2 dt$.

(c) Evaluate $\int_1^x [\![t^2]\!] \, t \, dt$.

15. Find the maximum and the minimum of the function

$$F(x) = \int_0^x \frac{2t + 1}{t^2 - 2t + 2} dt$$

in the interval $[-1,1]$.

16. Compute

(a) $$\int_{-\pi/4}^{\pi/4} \theta^{20}(\sin^{19}\theta - \sin^{17}\theta)\,d\theta \ .$$

(b) $$\int_{-2}^{2} \frac{x^{19} - x^{17} + x}{\cos^2 \frac{x}{2}} \ .$$

(c) $$\int_{-2}^{2} \sin^2 x \log \frac{3 + x}{3 - x}\,dx \ .$$

17. Prove that $\int_a^b f(x)dx = \int_a^b f(a + b - x)dx$.

18. (a) Compute

$$\lim_{\epsilon \sim 0^+} \int_{a\epsilon}^{b\epsilon} \frac{dx}{x} \ , \ b > a > 0 \ .$$

∧(b) Assume that f is continuous on $[0,\infty]$ and $\int_1^\infty \frac{f(x)}{x}dx$ exists. Show that

$$I = \int_0^\infty \frac{f(bx) - f(ax)}{x}dx = f(0)\log\frac{b}{a} \ .$$

∧19. By evaluating $I = \int_0^1 (1 - t^2)^n dx$, n a positive integer, in two different ways show that

$$\frac{\binom{n}{0}}{1} - \frac{\binom{n}{1}}{3} + \frac{\binom{n}{2}}{5} - \ldots + \frac{(-1)^n\binom{n}{n}}{2n + 1} = \frac{2^{2n}}{\binom{2n}{n}(2n + 1)} \ .$$

∧20. Let $\phi_0(x) = \phi(x)$ be a continuous function defined on $[0,x]$. Define $\phi_n(x)$ by

$$\phi_n(x) = \int_0^x \phi_{n-1}(y)dy \ , \ n = 1 \ , 2 \ , \ldots \ .$$

Show that

$$\phi_{n+1}(x) = \frac{1}{n!}\int_0^x \phi(y)(x - y)^n dy \ .$$

21. Prove that for all integers N

$$0 < \sum_{n=1}^{N} \frac{a}{a^2 + n^2} < \frac{\pi}{2} .$$

22. (a) Let L and M be linear operators and define the sum $L + M$ as the operator

$$L + M : u \longrightarrow L[u] + M[u] .$$

Verify that $L + M$ is linear.

(b) Show that linear operators satisfy the distributive laws:

$$L(M + N) = LM + LN$$
$$(L + M)N = LN + MN .$$

(c) Let f be any real-valued function. The multiplication operator $f\cdot$ is defined by

$$f \cdot [u] = f \cdot u .$$

Verify that $f\cdot$ is linear.

(d) For real-valued functions $a_n , a_{n-1} , \cdots , a_0$ show that the differential operator L of n-th order given by

$$L = a_n \cdot D^n + a_{n-1} \cdot D^{n-1} + \ldots + a_0$$

is linear.

23. (a) Prove for the second order linear differential operator L defined by $L = D^2 + p \cdot D + q$ that if u is a nontrivial solution of the homogeneous equation $L[u] = 0$, there is another solution of the form $u \cdot v$ where v' satisfies a linear differential equation $M[v'] = 0$ and M is at most of first order.

(b) Prove, in general, for the n-th order linear differential operator L defined by $L = \sum_{k=0}^{n} a_k(x) D_x^{\,k}$, $a_n(x) \neq 0$ that if u is a nontrivial solution of the homogeneous equation $L[u] = 0$, there is another solution of the form $u \cdot v$ where v' satisfies a linear differential equation $M[v'] = 0$ where M is at most of $(n-1)$-th order.

(c) Find an n-parameter family of solutions of the differential equations

$$L[u] = (D - a)^n u = 0 \ .$$

(d) Obtain the general solution of

$$(x^2 + 1)y'' + xy' - 4y = 0 \ .$$

(Hint: Try to find a particular polynomial solution.)

24. Two functions u and v are called linearly independent on an interval I if

$$\alpha u(x) + \beta v(x) = 0 \ , \text{ for all } x \text{ in } I \ ,$$

implies that α and β are both zero; i.e., if u and v are not proportional. Show that if two linearly independent solutions of a second order homogeneous linear equation exist in a neighborhood of x_0, then the initial value problem (13) of Section 10-8 can be solved.

25. Corresponding to the three classes of second order homogeneous linear equation with constant coefficients, Section 10-8, we found the following pairs of solutions $\{e^{\alpha x}, e^{\beta x}\}$ where $\alpha \neq \beta$, $\{e^{ax}, xe^{ax}\}$, and $\{e^{ax} \cos \omega x, e^{ax} \sin \omega x\}$ where $\omega \neq 0$.

Prove that each of these pairs of solutions is linearly independent.

26. Let u be any nontrivial solution of the homogeneous second order linear equation

$$L[u] = D^2 u + p \cdot Du + q \cdot u = 0 \ .$$

Obtain a second solution v in the form $v = u \cdot z$. Show that u and v are linearly independent.

27. Let u and v be any linearly independent solutions of the second order linear homogeneous equation. Verify that the Green's function is not affected if u and v replace the fundamental set (ϕ, ψ); namely

$$\begin{aligned} G(x,\xi) &= \frac{\psi(x)\phi(\xi) - \phi(x)\psi(\xi)}{\phi(\xi)\psi'(\xi) - \psi(\xi)\phi'(\xi)} \\ &= \frac{v(x)u(\xi) - u(x)v(\xi)}{u(\xi)v'(\xi) - v(\xi)u'(\xi)} \ . \end{aligned}$$

28. Discuss the solution of equations of the type $y'' + f(y) = 0$.

Appendix 6

EXISTENCE OF INTEGRALS

A6-1. Integration by Summation Techniques.

(1) Integral of a polynomial. In Section 6-4 we prove that integration is a linear operation, that the integral of a linear combination of functions is the same linear combination of their integrals:

$$\int_a^b [c_1f_1(x) + c_2f_2(x) + \dots + c_nf_n(x)]dx$$

$$= c_1\int_a^b f_1(x)dx + c_2\int_a^b f_2(x)dx + \dots + c_n\int_a^b f_n(x)dx .$$

In particular for a polynomial, we have

$$\int_a^b \sum_{r=0}^{n} c_rx^r \, dx = \sum_{r=0}^{n} c_r\int_a^b x^r \, dx .$$

In order to integrate a polynomial, then, it is sufficient to be able to integrate positive integral powers.

From the Corollary to Theorem 6-4b, we have

$$\int_a^b f(x)dx = \int_c^b f(x)dx - \int_c^a f(x)dx$$

provided that f is integrable over an interval containing the points a, b, c. In particular, for a polynomial we have

$$\int_a^b f(x)dx = \int_0^b f(x)dx - \int_0^a f(x)dx .$$

We need therefore consider only integrals of the type $\int_0^a f(x)dx$.

Consider, in particular, the integral of x^r over $[0,a]$. Since $0 \le x \le a$, the function x^r is increasing on the interval. We take a partition σ which subdivides the interval into n equal parts of length $h = \nu(\sigma) = \frac{a}{n}$. We form the upper sum U over σ using the maximum of x^r in each subinterval; thus

A6-1

(1)
$$U = \sum_{k=1}^{n} x_k^r(x_k - x_{k-1})$$
$$= \sum_{k=1}^{n} (kh)^r h$$
$$= h^{r+1} \sum_{k=1}^{n} k^r .$$

According to Equation (4) of Section A3-2 (ii) we have

$$k^r = \frac{k^{r+1} - (k - 1)^{r+1}}{r + 1} + P(k)$$

where P is a polynomial of degree $r - 1$. It follows that

(2)
$$U = \frac{h^{r+1}}{r + 1} \sum_{k=1}^{n} [k^{r+1} - (k - 1)^{r+1}] + Q(h)$$

where

(3)
$$Q(h) = h^{r+1} \sum_{k=1}^{n} P(k)$$

and P is a polynomial of degree $r - 1$.

We recognize the sum in (2) as telescoping (Section A3-2(ii)) and obtain

$$U = \frac{h^{r+1}}{r + 1}[n^{r+1} - 0] + Q(h) = \frac{(nh)^{r+1}}{r + 1} + Q(h) .$$

Since $nh = a$, we have

(4)
$$U = \frac{a^{r+1}}{r + 1} + Q(h) .$$

We can show that $Q(h)$ can be made closer to zero than any given error tolerance using only that the degree of $P(k)$ is at most $r - 1$. We set $P(k) = \sum_{i=1}^{r-1} p_i k^i$. Since $k \leq n$ it follows that

$$|P(k)| \leq \sum_{i=1}^{r-1} |p_i| k^i \leq \sum_{i=1}^{r-1} |p_i| n^i \leq \sum_{i=1}^{r-1} |p_i| n^{r-1} \leq n^{r-1} \sum_{i=1}^{r-1} |p_i| ,$$

In short, we have found

$$(5) \qquad |P(k)| \leq Cn^{r-1}$$

where the constant C is simply the sum of the absolute values of the coefficients of $P(x)$. Entering the result of (5) in (3), we have

$$
\begin{aligned}
(6) \qquad |Q(h)| &\leq h^{r+1} \sum_{k=1}^{n} |P(k)| \\
&\leq h^{r+1} \sum_{k=1}^{n} Cn^{r-1} \\
&\leq h^{r+1} \cdot n(Cn^{r-1}) \\
&\leq Ca^{r}\, h \ ,
\end{aligned}
$$

where again we use the fact that $nh = a$. It follows at once that $\lim_{h \to 0} Q(h) = 0$.

We could also form the lower sum L over σ by taking the minimum value of x^r as lower bound in each interval $[x_r, x_{r-1}]$. In this way we could obtain a result for L similar to (4) and so prove

$$(7) \qquad \int_0^a x^r dx = \frac{a^{r+1}}{r+1} \ ;$$

the details are left to the reader.

(ii) A cosine integral. Let us attempt to find the integral of $\cos x$ over $[0,a]$ where we suppose $a < \pi$ so that $\cos x$ is decreasing on the interval. We take a subdivision of the interval into n equal parts of length $h = \frac{a}{n}$. Setting

$$x_k = kh \ , \qquad (k = 1, 2, \ldots, n) \ ,$$

we obtain a lower sum L over σ

$$(1) \qquad L = \sum_{k=1}^{n} (\cos x_k)(x_k - x_{k-1}) = h \sum_{k=1}^{n} \cos kh$$

and an upper sum U over σ

$$U = h \sum_{k=1}^{n} \cos(k - 1)h$$

$$= L + h[1 - \cos a] .$$

From Equation (7) of Section A3-2(ii), on setting

$$\cos \frac{n(a + 1)}{2} \sin \frac{na}{2} = \frac{1}{2}[\sin(n + \tfrac{1}{2})a - \sin \tfrac{a}{2}]$$

we obtain

$$(2) \qquad \sum_{k=1}^{n} \cos kz = u(n) - u(0) = \frac{\sin(n + \frac{1}{2})z}{2 \sin \frac{1}{2} z} - \frac{1}{2} .$$

Equation (2) permits us to evaluate the limit of the lower sum given in Equation (1):

$$\lim_{h \sim 0} L = \lim_{h \sim 0} \frac{\frac{1}{2}h}{\sin \frac{1}{2} h} \sin(a + \tfrac{1}{2}h) - \frac{h}{2} .$$

Using the fact that $\lim_{u \sim 0} \frac{\sin u}{u} = 1$ we have

$$\lim_{h \sim 0} L = \sin a .$$

Since the difference between L and U has the limit 0, we conclude that

$$\int_0^a \cos x \, dx = \sin a .$$

Exercises A6-1

1. In subsection (i) the text states that it follows "at once" from the inequality (6) that

$$\lim_{h \sim 0} Q(h) = 0 .$$

Actually, what theorems on limits are being used?

2. Show simply, without repeating the argument of the text, that the lower sum L over σ, $L = \sum_{k=1}^{n} x_{k-1}{}^{r}(x_{k-1} - x_k)$ also has the limit (7).

3. Employ Equation (8) of Section A3-2(ii) to obtain $\int_0^a \sin x \, dx$ for $0 < a \leq \frac{\pi}{2}$.

A6-2. Existence of the Integral.

The purpose of this section is to establish necessary and sufficient conditions for the existence of the integral of a function f over $[a,b]$. Recall that the integral is defined as the unique separation number between the upper and lower sums. We need first to establish that the upper and lower sums are in fact separated: that every lower sum is less than or equal to every upper sum. If it is possible to find an upper sum and a lower sum closer together than any given fixed tolerance ϵ, than by Lemma A1-5 there exists a unique separation number, a number I which is the integral of f over $[a,b]$.

Lemma A6-2a. Let f be a function defined and bounded on $[a,b]$. For any fixed partition σ of $[a,b]$, each upper sum U over σ is greater than or equal to each lower sum L over σ.

Proof. We recall that the partition σ is simply a set of points of $[a,b]$ which includes the endpoints a and b. To construct upper and lower sums, the points of σ are arranged in increasing order; i.e.,

$$a = x_0 < x_1 < x_2 < \dots < x_n = b .$$

An upper sum U is defined as

$$U = \sum_{k=1}^{n} M_k(x_k - x_{k-1})$$

where $f(x) \leq M_k$ on $[x_{k-1},x_k]$, a lower sum as

$$L = \sum_{k=1}^{n} m_k(x_k - x_{k-1})$$

where $f(x) \geq m_k$ on $[a_{k-1},x_k]$. Thus $m_k \leq M_k$ and term-for-term

$$m_k(x_k - x_{k-1}) \leq M_k(x_k - x_{k-1})$$

from which this lemma follows.

It is necessary to find a means of comparing upper and lower sums for any two partitions σ_1 and σ_2. For this purpose we introduce the joint partition $\sigma = \sigma_1 \cup \sigma_2$ which consists of all points of the two partitions taken together. Let U_1 be any upper sum over σ_1 and L_2 any lower sum over σ_2. We shall show that U_1 is an upper sum for the joint subdivision

σ and that L_2, similarly, is a lower sum for σ. The result we seek will then follow from the preceding lemma.

Lemma A6-2b. For any partitions σ_1 and σ_2 of $[a,b]$ and any upper and lower sums U_1, L_2, over the respective subdivisions,

$$U_1 \geq L_2 .$$

Proof. Let x_{k-1}, x_k be a pair of consecutive points of subdivision from σ_1, $(k = 1, 2, \ldots, n)$. There may be points of the subdivision σ_2 in the open interval (x_{k-1}, x_k), say, $u_1, \ldots, u_{p-1}$ with $x_{k-1} < u_1 < u_2 < \ldots < u_{p-1} < x_k$. Setting $u_0 = x_{k-1}$ and $u_p = x_k$ we see that the set $\{u_i : i = 0, \ldots, p\}$ is a partition of $[x_{k-1}, x_k]$. Further since M_k and m_k are upper and lower bounds for $f(x)$ in all of $[x_{k-1}, x_k]$ they are bounds for $f(x)$ in each of the subintervals $[u_{i-1}, u_i]$, $i = 0, 1, 2, \ldots, p$, (see Figure A6-2). If we form the upper sum U_k^* over the partition of $[x_{k-1}, x_k]$ using the upper bound M_k we have

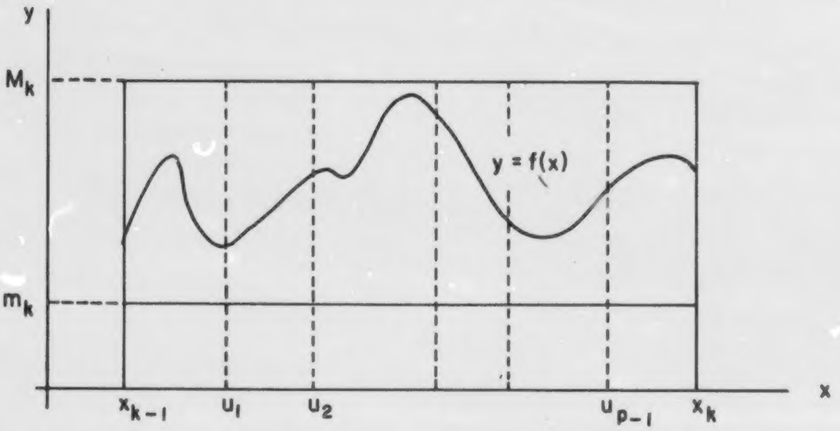

Figure A6-2

$U_k^* = \sum_{i=1}^{p} M_k(u_i - u_{i-1}) = M_k \sum_{i=1}^{p} (u_i - u_{i-1}) = M_k(x_k - x_{k-1})$. Thus the upper sum $U_1 = \sum_{k=1}^{n} U_k^*$ for the partition σ_1 is also an upper sum for σ. Similarly L_2 is a lower sum for both σ_2 and σ. It follows from Lemma A6-2a that

$$L_2 \leq U_1 .$$

Corollary. If for any partitions σ_1 and σ_2 of $[a,b]$ there exist an upper sum U_1 over σ_1 and a lower sum L_2 over σ_2 satisfying

$$U_1 - L_2 < \epsilon ,$$

then there exists a partition σ which has upper and lower sums U and L satisfying

$$U - L < \epsilon .$$

Proof. Take $\sigma = \sigma_1 \cup \sigma_2$. Since U_1 and L_2 are upper and lower sums for the joint partition, the result is immediate.

THEOREM 6-3a. Let f be a bounded function on $[a,b]$. If for every positive ϵ there exists a partition σ of $[a,b]$ and lower and upper sums L and U over σ which differ by less than ϵ, then there exists a number I which is the integral of f over $[a,b]$ .. Conversely, if f is integrable over $[a,b]$, then there exist a partition σ and lower and upper sums L and U over σ such that $U - L < \epsilon$.

Proof. From Lemma A6-2b every lower sum is less than or equal to each upper sum. If for every $\epsilon > 0$ there exist lower and upper sums L and U satisfying $U - L < \epsilon$, then by Lemma A1-5 the number separating the set of lower sums from the set of upper sums is unique. By Definition 6-3 this separation number is the integral of f over $[a,b]$.

Conversely, if f is integrable, that is, if the integral of f over $[a,b]$ exists, then by Definition 6-3 the separation number between lower and upper sums in unique. It follows from the converse statement in Lemma A1-5 that there exist lower and upper sums, not necessarily over the same partition, say L_1 over σ_1 and U_2 over σ_2 for which $U_2 - L_1 < \epsilon$. From the corollary to Lemma A6-3b, we conclude that there exists a single partition σ having upper and lower sums U and L for which $U - L < \epsilon$.

Next we prove a useful corollary to Theorem 6-2a.

Lemma A6-2c. If f is integrable over $[a,b]$ then f is integrable over any subinterval $[\alpha,\beta]$.

Proof. There exists a partition σ of $[a,b]$ for which $U - L < \epsilon$ where U and L denote upper and lower sums over σ . We may assume α and β are points of σ , for if they were not so originally they could be introduced without affecting the values of U and L (see the proof of Lemma A6-3b). With α and β included in σ , it follows that σ contains a partition σ' of $[\alpha,\beta]$. Now in the sum

$$U - L = \sum (M_k - m_k)(x_k - x_{k-1})$$

all terms are nonnegative. If we let U' and L' denote those parts of the sums U and L which are taken over σ' , it follows that

$$U' - L' \leq U - L < \epsilon .$$

According to Theorem 6-3a, the function f is integrable over $[\alpha,\beta]$.

Our method of defining the integral has avoided certain analytical complications associated with the definition of the integral as a limit of Riemann sums. Some appreciation for the analytical difficulties may be gleaned from the following discussion.

In order to establish the connection between upper and lower sums and the Riemann sums of the text (Section 6-3(iii)) we need one further result.

Lemma A6-2d. If f is integrable over $[a,b]$ then for all sufficiently fine subdivisions σ there exist upper and lower sums closer than any fixed tolerance ϵ .

In other terms, for every $\epsilon > 0$ there exists a $\delta > 0$ such that for every partition σ with norm $\nu(\sigma)$ less than δ , there exist both an upper sum U and a lower sum L over σ for which $U - L < \epsilon$.

Proof. From the existence of the integral and the Corollary to Lemma A6-2b we know that there exists a partition $\overline{\sigma} = \{x_0, x_1, \ldots, x_n\}$ with upper and lower sums $\overline{U}$ and $\overline{L}$ satisfying $\overline{U} - \overline{L} < \overline{\epsilon}$. We let $\overline{M}_k$ and $\overline{m}_k$ denote upper and lower bounds for f in the subinterval $[x_{k-1}, x_k]$, and let M and m denote upper and lower bounds in the entire interval $[x_0, x_n]$.

Let $\sigma = \{u_0, u_1, \ldots, u_p\}$ be any partition of $[a,b]$. We separate the case for which a subinterval $[u_{i-1}, u_i]$ contains points of $\overline{\sigma}$ in its interior from the case in which it does not.

If $[u_{i-1}, u_i]$ contains a point of $\overline{\sigma}$ in its interior we choose the overall bounds M and m of f on $[a,b]$ as bounds for the function in the subinterval. Since there are at most the $n - 1$ points $x_1, x_2, \ldots, x_{n-1}$ which could be interior points of intervals of σ, there can be no more than $n - 1$ such intervals containing points of $\overline{\sigma}$. We form the partial upper sum U_1 and partial lower sum L_1 over these intervals and find

$$U_1 - L_1 \leq (n - 1)(M - m)\nu(\sigma) \,. \tag{1}$$

If $[u_{i-1}, u_i]$ does not contain a point of $\overline{\sigma}$ in its interior, then $[u_{i-1}, u_i]$ must lie wholly within an interval $[x_{k-1}, x_k]$ of $\overline{\sigma}$. We take as upper and lower bounds for f on $[u_{i-1}, u_i]$ the bounds M_k and m_k for f on the interval $[x_{k-1}, x_k]$. For all the intervals of σ contained in $[x_{k-1}, x_k]$ the total contribution to the difference between the upper and lower sums is less than or equal to $(M_k - m_k)(x_k - x_{k-1})$. Forming the partial upper sum U_2 and partial lower sum L_2 over all those intervals of σ which contain no points of $\overline{\sigma}$ we find

$$\begin{aligned} U_2 - L_2 &\leq \sum_{k=1}^{n} (M_k - m_k)(x_k - x_{k-1}) \\ &\leq \overline{U} - \overline{L} \\ &< \overline{\epsilon} \,. \end{aligned} \tag{2}$$

For the complete upper sum $U = U_1 + U_2$ and complete lower sum $L = L_1 + L_2$ over σ we have

$$\begin{aligned} U - L &= (U_1 + U_2) - (L_1 + L_2) \\ &= (U_1 - L_1) + (U_2 - L_2) \\ &\leq (n - 1)(M - m)\nu(\sigma) + \overline{\epsilon} \,. \end{aligned}$$

We can make the difference $U - L$ less than ϵ by making each term in the last expression less than $\frac{\epsilon}{2}$. It is sufficient, then, to take $\bar{\epsilon} = \frac{\epsilon}{2}$ and $\delta = \frac{\epsilon}{2[(n-1)(M-m)+1]}$ (the denominator being chosen to guard against the possibility that $(n-1)(M-m) = 0$). For $\nu(\sigma) < \delta$ the lemma is established.

It is now easy to prove that integrability implies Riemann integrability.

THEOREM 6-3c. The value I is the integral of f over $[a,b]$, in the sense of Definition 6-3, if and only if it is the limit of Riemann sums

$$I = \lim_{\nu(\sigma)\sim 0} R \,. \tag{3}$$

Proof. As before, consider a partition $\sigma = \{x_0, x_1, \ldots, x_n\}$ with $m_k \leq f(x) \leq M_k$ on $[x_{k-1}, x_k]$. For any particular value ξ_k in the interval $[x_{k-1}, x_k]$ we have $m_k \leq f(\xi_k) \leq M_k$, whence,

$$\sum_{k=1}^{n} m_k(x_k - x_{k-1}) \leq \sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}) \leq \sum_{k=1}^{n} M_k(x_k - x_{k-1})$$

or

$$L \leq R \leq U$$

for all Riemann sums and all lower and upper sums L and U over σ. Using Lemma A6-2d we can obtain upper and lower sums U and L for which the difference $U - L$ is smaller than any given positive ϵ provided the partition is fine enough; i.e., $\nu(\sigma) < \delta$ for a suitable positive δ. We have simultaneously

$$L \leq I \leq U$$

and

$$L \leq R \leq U$$

for all Riemann sums on σ. It follows that

$$|R - I| \leq U - L < \epsilon \,.$$

Thus we have satisfied the criterion that I is the appropriate limit of Riemann sums.

As a research problem complete the proof of the theorem (Exercises A6-2, No. 3).

Exercises A6-2

1. Let f be a function which takes on a maximum and minimum on every closed interval (e.g., f could be a continuous function, or monotone). Let $U^*(\sigma)$ and $L^*(\sigma)$ be the upper and lower Riemann sums obtained by using the maximum and minimum values of $f(x)$ as the appropriate bounds in each interval of the subdivision.

   Let σ_1 and σ_2 be any partitions of $[a,b]$. Prove for the joint subdivision $\sigma = \sigma_1 \cup \sigma_2$ that

   $$U^*(\sigma_1) \geq U^*(\sigma) \geq L^*(\sigma) \geq L^*(\sigma_2)$$

   In other terms, by adding new points to a subdivision we may reduce the difference between the upper and lower Riemann sums, and we cannot increase it.

2. Show that if f is Riemann integrable (Section 6-4) over $[a,b]$, then f is bounded on $[a,b]$.

∧3. Let f be integrable over $[a,b]$ and let R denote a Riemann sum corresponding to a partition σ of $[a,b]$. We have proved (Theorem 6-3c) that if f has an integral I then

   $$I = \lim_{\nu(\sigma)\sim 0} R .$$

   Prove conversely that if the limit of the Riemann sums exists then it is the integral of f over $[a,b]$. (Hint: Show first for any partition and positive ϵ that there exist at least one Riemann sum R and one upper sum U over σ such that $U - R < \epsilon$.)

4. Consider the function f defined on $[0,1]$ by

   $$f(x) = \begin{cases} 0, & x \text{ irrational} \\ 1, & x \text{ rational} \end{cases}$$

   Prove that the integral of f does not exist.

5. Consider the function f defined on $[0,1]$ by

   $$f(x) = \begin{cases} 0, & x \text{ irrational} \\ \frac{1}{t}, & x \text{ rational, } x = \frac{s}{t} \text{ in lowest terms.} \end{cases}$$

   Prove that the integral of f over $[0,1]$ exists and find its value.

∧6. Give an example of a nonintegrable function fg where f and g are each integrable.

Appendix 7

INTEGRABILITY OF CONTINUOUS FUNCTIONS

A7-1. Covers of Closed Intervals.

In order to prove the integrability of continuous functions we introduce the idea of a cover of an interval. A set C of open intervals is said to be a cover of an interval I if for every x in I there is a member of C which contains, or covers x. If f is continuous on the closed interval $[a,b]$ then for every positive ϵ each point x in $[a,b]$ has a neighborhood $N(x)$ with the property that

$$|f(u) - f(x)| < \epsilon$$

for all u in $N(x)$ and in $[a,b]$. For each positive ϵ, the set of such neighborhoods is a cover of the interval $[a,b]$. This cover is an infinite set of neighborhoods. The remarkable property which enables us to prove the general integrability of continuous functions is that this infinite set of neighborhoods can be replaced by a subset which is also a cover of $[a,b]$.

THEOREM A7-1. The Heine-Borel Principle. Every cover of a closed interval contains a cover consisting of finitely many open sets.

Proof. We shall use the Nested Interval Principle (Section A1-5) to prove this result. Let C be a set of open intervals which cover $[a,b]$. We suppose that no finite subset of C is a cover of $[a,b]$ and seek a contradiciton. The two half-intervals $[a,\frac{1}{2}(a + b)]$ and $[\frac{1}{2}(a + b),b]$ cannot both have finite covers within C for on combining the two covers we should obtain a finite cover of $[a,b]$. Thus at least one of the half-intervals has no finite cover. Let $[a_1,b_1]$ be a half-interval which has no finite cover. Again, the half-intervals $[a_1,\frac{1}{2}(a_1 + b_1)]$ and $[\frac{1}{2}(a_1 + b_1),b_1]$ cannot both have finite covers. We can then choose a half-interval without a finite cover and denote it by $[a_2,b_2]$. In general, if we have an interval $[a_k,b_k]$ without a finite cover, we denote by $[a_{k+1},b_{k+1}]$ one of the half-intervals of $[a_k,b_k]$ which has no finite cover.

The intervals $[a_k, b_k]$ of the preceding construction are nested:

$$[a_{k+1}, b_{k+1}] \subset [a_k, b_k] .$$

It follows from the Nested Interval Principle that there is at least one real number s in all these intervals,

$$a_k < s \leq b_k , \qquad (k = 1 , 2 , 3 , \ldots).$$

Moreover,

$$\begin{aligned} b_k - a_k &= \frac{1}{2}(b_{k-1} - a_{k-1}) \\ &= \frac{1}{4}(b_{k-2} - a_{k-2}) = \ldots = \frac{1}{2^{k-1}}(b_1 - a_1) , \\ &= \frac{b - a}{2^k} , \end{aligned}$$

so that the difference $b_k - a_k$ is made less than any given tolerance for sufficiently large k . It follows by Lemma A1-5, that the number s separating the set of lower endpoints from the set of upper endpoints is determined uniquely.

Since s is a point of $[a,b]$, it is covered by some open interval in C , say (u,v) . Since $u < s < v$ it follows that $\text{Min}\{s - u , v - s\}$ is positive. If $\epsilon = \text{Min}\{s - u , v - s\}$, then $b_k - a_k = \frac{b - a}{2^k} < \epsilon$ for any sufficiently large k and $[a_k, b_k]$ is contained in (u,v) . It was asserted that $[a_k, b_k]$ had no finite cover in C , but now we find that it can be covered by the one interval (u,v) . This is the contradiction we sought.

Exercises A7-1

1. Show that the Heine-Borel Principle fails for the interval $[1, \sqrt{2})$; that is, find a cover C of $[1, \sqrt{2})$ such that no finite subset of C is a cover of $[1, \sqrt{2})$.
2. Prove that the Heine-Borel Principle fails for open intervals; that is, find a cover C of an open interval such that no finite subset of C is a cover.
3. Complete the demonstration that the Heine-Borel Principle is equivalent to the Separation Axiom; that is, show in an ordered field that the principle implies the axiom.

4. State and prove the converse of the Heine-Borel Principle.
5. Since the Separation Axiom fails for the field of rational numbers, so also must the Heine-Borel Principle. State the would-be Heine-Borel Principle for rational numbers and show by example that it is not valid.

A7-2. The Integral of a Continuous Function.

Using the Heine-Borel Principle we derive the basic result:

THEOREM A7-2. If f is continuous on the interval $[a,b]$, then f is integrable over $[a,b]$.

Proof. Let x be any point of $[a,b]$. For a given positive ϵ let $N(x)$ be a neighborhood of x for which

$$|f(u) - f(x)| < \epsilon \tag{1}$$

whenever u is in $N(x)$. The neighborhood $N(x)$ consists of all the points u satisfying

$$|u - x| < \delta(x)$$

for some value $\delta(x)$. We shall also make use of the neighborhood with radius $\frac{1}{2}\delta(x)$,

$$\overline{N}(x) = \{u : |u - x| < \tfrac{1}{2}\delta(x)\} .$$

The set of neighborhoods $\overline{N}(x)$ for x in $[a,b]$ is a cover of $[a,b]$. From the Heine-Borel Principle it follows that there is a finite subset or neighborhoods $\overline{N}(x_i)$, $(i = 1 , 2 , \dots , n)$ which cover the interval. If δ_i is the radius of $N(x_i)$, then $\frac{1}{2}\delta_i$ is the radius of $\overline{N}(x_i)$. We set

$$\delta = \tfrac{1}{2}\,\mathrm{Min}\{\delta_i\} .$$

Now let $\sigma = \{u_0 , u_1 , u_2 , \dots , u_n\}$ be any partition with norm $\nu(\sigma) < \delta$. In each subinterval $[u_{k-1} , u_k]$ we shall find upper and lower bounds for $f(x)$,

$$m_k \leq f(x) \leq M_k$$

which differ by at most a fixed multiple of ϵ .

Let x by any point of $[u_{k-1},u_k]$. Since $u_k - u_{k-1} < \delta \leq \frac{1}{2}\delta_i$ it follows that $|x - u_k| < \frac{1}{2}\delta_i$. Now let $\overline{N}(x_i)$ be a covering interval of u_k . Since u_k is covered by an open interval $\overline{N}(x_i)$ we have $|u_k - x_i| < \frac{1}{2}\delta_i$. Consequently

$$|x - x_i| < \delta_i ;$$

that is, x is a point of $N(x_i)$. We conclude that (1) is satisfied:

$$|f(x) - f(x_1)| < \epsilon$$

or

$$f(x_1) - \epsilon < f(x) < f(x_1) + \epsilon .$$

Taking $M_k = f(x_1) + \epsilon$, $m_k = f(x_1) - \epsilon$ we have upper and lower bounds satisfying

$$M_k - m_k = 2\epsilon .$$

It follows for the difference between the corresponding upper and lower sums that

$$\begin{aligned} \sum_{k=1}^{n} M_k(u_k - u_{k-1}) - \sum_{k=1}^{n} m_k(u_k - u_{k-1}) &= \sum_{k=1}^{n} (M_k - m_k)(u_k - u_{k-1}) \\ &= 2\epsilon \sum_{k=1}^{n} (u_k - u_{k-1}) \\ &= 2\epsilon(b - a) . \end{aligned}$$

Since we can find an upper sum and lower a lower sum which differ by less than any prescribed tolerance, the integrability of f is proved.

Exercises A7-2

1. A function is said to be of bounded variation on $[a,b]$ if there exists a bound M such that for all partitions $\sigma = \{x_0, x_1, x_2, \ldots, x_n\}$ of $[a,b]$.

$$\lambda(\sigma) = \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \leq M .$$

   (a) Prove if the function f is monotone on $[a,b]$ then f is of bounded variation on $[a,b]$.

   (b) Prove if f is a function of bounded variation then f can be represented as a sum $f = g + h$ where g is weakly increasing and h is weakly decreasing. Prove conversely, if f can be represented as such a sum of monotone functions, then f is a function of bounded variation.

2. The preceding result enables us to prove that not all integrable functions are linear combinations of monotone functions. Consider the function defined by

$$f(x) = \begin{cases} x \cos \frac{\pi}{x}, & x \neq 0 \\ 0, & x = 0 . \end{cases}$$

Prove that f is continuous, therefore integrable on $[0,1]$. Prove also that f is not of bounded variation on $[0,1]$.

3. Prove if f is continuous on the closed interval $[a,b]$ and has a bounded derivative in the interior (a,b) , then f is of bounded variation on $[a,b]$.

4. Prove that if the function f is of bounded variation on $[a,b]$ and if $a \leq c < d \leq b$, then f is of bounded variation on $[c,d]$.

5. The concept of length of a curve, like that of area, is not defined in general by the methods of elementary geometry. By analogy with the concept of integral, it is natural to attempt to express the length of a curve as a limit of the lengths of polygonal approximations. Let a curve be given as the graph of a given function f on $[a,b]$. Given a partition $\sigma = \{x_0, x_1, \ldots, x_n\}$ of $[a,b]$ we construct an

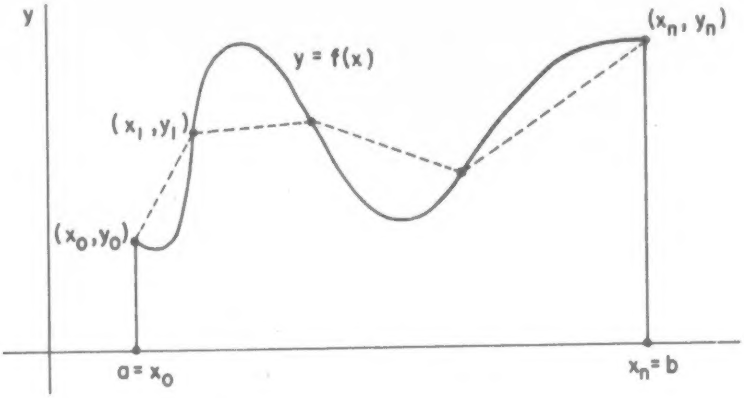

Figure A7-2

inscribed polygon to the graph of f by joining the successive points (x_k,y_k) where $y_k = f(x_k)$, for $k = 0, 1, 2, \ldots, n$ (Figure A7-2). Let $\mathcal{L}$ denote the length of the graph and $L(\sigma)$ the length of the inscribed polygon. We have

$$L(\sigma) = \sum_{k=1}^{n} \sqrt{(x_k - x_{k-1})^2 + (y_k - y_{k-1})^2}$$

where the general term in the sum is the length of the straight-line segment joining the points (x_{k-1},y_{k-1}) and (x_k,y_k). Intuitively, the straight path is the shortest path between the two points, so that the path along the graph of f is never shorter than the segment joining the two points. We must then have $L(\sigma) \leq \mathcal{L}$ for all partitions σ. We can only estimate $\mathcal{L}$ from below by using inscribed polygons. Furthermore, there is no obvious general way of estimating $\mathcal{L}$ from above. For this reason we define $\mathcal{L}$ as the least upper bound of the lengths $L(\sigma)$ of inscribed polygons, provided such a bound exists. If the length $\mathcal{L}$ exists we say that f is rectifiable over $[a,b]$. Prove that a necessary and sufficient condition for f to be rectifiable over $[a,b]$ is that f be of bounded variation on $[a,b]$.

6. Let α be a point of the domain of f for which every deleted neighborhood contains other points of the domain. The function f is said to be increasing at the point α if there exists some neighborhood wherein

   (i) $$x < \alpha \Longrightarrow f(x) < f(\alpha)$$

   and

   (ii) $$x > \alpha \Longrightarrow f(x) > f(\alpha)$$

   for those points x in the domain of f. Show that if f is increasing at every point of the interval (a,b) then f is increasing on (a,b).

7. By the Heine-Borel Principle prove that if f is continuous on $[a,b]$ then f is bounded on $[a,b]$, (Theorem A4-1).

Appendix 8

ANALYTICAL DEFINITION OF THE CIRCULAR FUNCTIONS

In Section 8-5 we defined the sine and cosine functions as solutions of the differential equation

(1) $$D^2f + f = 0$$

where $\sin : x \longrightarrow \phi(x)$ satisfies the initial condition

(2) $$\phi(0) = 0 \ , \ \phi'(0) = 1$$

and $\cos : x \longrightarrow \psi(x)$ satisfies the initial condition

(3) $$\psi(0) = 1 \ , \ \psi'(0) = 0 \ .$$

We wish to prove the results that the differential equations define the two circular functions for all real values of x and that these functions are periodic with period 2π.

We note first that the inverse g of ϕ, is defined by the integral

(4) $$x = g(u) = \int_0^u \frac{1}{\sqrt{1 - t^2}}\, dt$$

for all values u in the open interval $-1 < u < 1$. From this fact we may conclude only that ϕ is defined on some neighborhood of the origin by

(5) $$\phi : g(u) \longrightarrow u \qquad (|u| < 1) \ .$$

Wherever ϕ is defined, we define ψ by

(6) $$\psi : x \longrightarrow \phi'(x) \ .$$

Our first problem is to extend these definitions to the domain of all real numbers. Once we have shown the existence of the solutions ϕ and ψ of the differential equation (1) on the domain of all real numbers we are free to employ the addition theorem (Exercises 8-5) without restriction since the sum of two numbers in the domain will again be in the domain. Using the addition theorems, we shall have no difficulty in establishing the periodicity of the functions.

Let x_0 be any real number. It is easily verified that

(7) $$f : x \longrightarrow a\psi(x - x_0) + b\phi(x - x_0)$$

is a solution of the differential equation (1) which satisfies the initial condition

$$(8) \qquad f(x_0) = a \ , \ f'(x_0) = b \ .$$

Furthermore, (7) is the only solution satisfying the condition (8) by exactly the same argument as that of the uniqueness theorem (Theorem 8-5b) for the case $x_0 = 0$.

Now let $\xi = g(\eta)$ where $0 < \eta < 1$ so that ϕ and ψ are defined by (5) and (6) on the closed interval $-\xi \leq x \leq \xi$. (Here we employ the symmetries of the two functions, Exercises 8-5, No. 10). In (7) and (8) we take $x_0 = \xi$, $a = \phi(\xi) = \eta$ and $b = \phi'(\xi)$. Thus the function ϕ satisfies the same initial conditions as f at $x_0 = \xi$ and therefore by the uniqueness theorem must coincide with f where the domains of the two functions overlap. The function f is therefore a natural extension of ϕ . The domain of ϕ includes the interval $[-\xi,\xi]$ and the domain of f includes the interval $[x_0 - \xi \ , x_0 + \xi] = [0,2\xi]$. The intersection of two intervals is the interval $[0,\xi]$. We introduce the function

$$\hat{\phi} : x \longrightarrow \begin{cases} \phi(x) \ , & x \in [-\xi,\xi] \\ \phi(x) = f(x) \ , & x \in [0,\xi] \\ f(x) \ , & x \in [\xi,2\xi] \ . \end{cases}$$

Clearly $\hat{\phi}$ satisfies the differential equation (1) on the interval $[-\xi,2\xi]$ and the initial condition (2). Finally we define the extension $\hat{\psi}$ of ψ on the interval $[-\xi,2\xi]$ by

$$\hat{\psi} : x \longrightarrow \hat{\phi}'(x) \ .$$

In order to keep the notation simple we no longer distinguish between the extended functions $\hat{\phi}$ and $\hat{\psi}$ and the original functions ϕ and ψ ; this cannot cause any confusion since the extensions are uniquely determined.

We may now repeat the procedure to extend the two functions further. Since ϕ and ψ are defined on the interval $[-\xi,2\xi]$ we may introduce the solution (7) of the differential equation which satisfies the same condition (8) at $x_0 = 2\xi$ as the function ϕ ; that is, we take $a = \phi(2\xi)$ and $b = \phi'(2\xi) = \psi(2\xi)$. The function f is defined on the interval $[x_0 - \xi \ , x_0 + 2\xi] = [\xi,4\xi]$ which overlaps the domain of ϕ on the interval $[\xi,2\xi]$. We extend the definitions of the functions ϕ and ψ by setting $\phi = f$ and $\psi = f'$ on the domain of f .

We proceed recursively. Once we have defined ϕ and ψ on the interval $[-\xi, 2^k\xi]$ we introduce the solution f of the differential equation defined by (7) and (8) where $x_0 = 2^k\xi$, $a = \phi(x_0)$, $b = \psi(x_0)$. The function f is then defined on the interval $[x_0 - \xi , x_0 + 2^k\xi] = [(2^k - 1)\xi , 2^{k+1}\xi]$ which overlaps the domain of ϕ on the interval $[(2^k - 1)\xi , 2^{k+1}\xi]$. By the uniqueness theorem, the functions ϕ and f are the same on the common part of their domains. We extend the definitions ϕ and ψ to the entire interval $[-\xi , 2^{k+1}\xi]$ by setting $\phi = f$ and $\psi = f'$ on the domain of f .

Given any positive real number x , there exists a value of k such that $2^k\xi > x$. It follows that $\phi(x)$ and $\psi(x)$ are uniquely defined for every positive value x . A similar construction may be used to define $\phi(x)$ and $\psi(x)$ for all negative values of x (see Exercises A8, No. 1). In this way we extend the definitions of the functions to all real values.

To show that the functions ϕ and ψ are periodic we must find an appropriate way to introduce the number π and to exhibit its relation to the period. We define the number π by the relation $\frac{\pi}{4} = g\left(\frac{1}{\sqrt{2}}\right)$, that is, from (4),

$$\frac{\pi}{4} = \int_0^{1/\sqrt{2}} \frac{1}{\sqrt{1 - t^2}}\, dt .$$

Equivalently, we have

$$\phi\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} . \tag{9}$$

Since $[\phi(x)]^2 + [\psi(x)]^2 = 1$ we conclude that $[\psi(\frac{\pi}{4})]^2 = \frac{1}{2}$ or that $\psi(\frac{\pi}{4}) = \pm \frac{1}{\sqrt{2}}$. We wish to determine the correct sign.

From (4) we know that ϕ is increasing and nonnegative on the interval $[0,\frac{\pi}{4}]$. Since $\psi' = -\phi$ (Exercises 8-5, No. 8) and is positive on $[0,\frac{\pi}{4}]$, then ψ' is decreasing on the interval. Since $\phi(x)$ can not attain the value 1 in $[0,\frac{\pi}{4}]$ it follows that $\psi(x)$ cannot be zero at any point of the interval. Observing that $\psi(0)$ is positive and that ψ is continuous we conclude that $\psi(\frac{\pi}{4}) > 0$, hence

$$\psi\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} . \tag{10}$$

We leave as an exercise the proofs of the details of this argument.

Next we insert the results (9) and (10) in the "double-angle" formulas (Section A2-5, (9) and (10)) to obtain successively

$$\begin{cases} \phi(\frac{\pi}{2}) = 2\,\phi(\frac{\pi}{4})\psi(\frac{\pi}{4}) = 1 \\ \psi(\frac{\pi}{2}) = [\psi(\frac{\pi}{4})]^2 - [\phi(\frac{\pi}{4})]^2 = 0 \end{cases}$$

$$\begin{cases} \phi(\pi) = 2\,\phi(\frac{\pi}{2})\psi(\frac{\pi}{2}) = 0 \\ \psi(\pi) = [\psi(\frac{\pi}{2})]^2 - [\phi(\frac{\pi}{2})]^2 = -1 \end{cases}$$

whence, finally

$$(11) \qquad \begin{cases} \phi(2\pi) = 2\,\phi(\pi)\psi(\pi) = 0 \\ \psi(2\pi) = [\psi(\pi)]^2 - [\phi(\pi)]^2 = 1 \end{cases}$$

The periodicity of the sine function now follows immediately from (11) upon taking $x_0 = 2\pi$, $a = 0$ and $b = 1$ in (7) and (8). We have

$$(12) \qquad f(x) = \phi(x - 2\pi)$$

where f satisfies exactly the same initial conditions at $x_0 = 2\pi$ as ϕ . It follows from the uniqueness theorem that $f = \phi$ and from (12) that

$$(13) \qquad \phi(x) = \phi(x - 2\pi) .$$

From (13) we see that ϕ is periodic with period 2π . To complete the proof we differentiate in (13) to obtain the same result for ψ .

We have used the symbols ϕ and ψ throughout instead of the familiar sin and cos in order to avoid the possibility of a logical slip through the inadvertent assumption without proof of one of the well known properties of the circular functions. Now that these properties have been established we shall return to the customary notation.

Exercises A8

1. (a) Employing the methods of the text, extend the definitions of $\sin : x \longrightarrow \phi(x)$ and $\cos : x \longrightarrow \psi(x)$ given by (4), (5), and (6) to all negative values of x .

   (b) Use the fact that the functions ϕ and ψ have been defined for $x \geq -\xi$, where $0 < \xi < 1$, to extend their domain to include the set of all negative numbers.

2. Prove:

   (a) $\phi(\frac{\pi}{2} - x) = \psi(x)$.

   (b) $\psi(\frac{\pi}{2} - x) = \phi(x)$.

   (c) $\phi(\pi - x) = \phi(x)$.

   (d) $\psi(\pi - x) = -\psi(x)$.

3. Show:

   (a) $\phi(\frac{\pi}{6}) = \frac{1}{2}$.

   (b) $\psi(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$.

   (c) $\phi(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$.

   (d) $\psi(\frac{\pi}{3}) = \frac{1}{2}$.

4. Show:

   (a) ϕ has no positive period less than 2π .

   (b) the function $\tau : x \longrightarrow \frac{\phi(x)}{\psi(x)}$ is periodic with period π .

5. Derive the formulas $\phi(z + 2\pi) = \phi(z)$ and $\psi(z + 2\pi) = \psi(z)$

   (a) from (13) .

   (b) directly from (11) and the addition theorems.

6. Prove that if t is the arc length of the curve $y = \sqrt{1 - x^2}$ between $x = 0$ and $x = a$ then $\phi(t) = a$ and $\psi(t) = \sqrt{1 - a^2}$.

Appendix 9

THE STORY ABOUT AL

Under a ruling by the Director of SMSG, the story about Al may be disseminated only by word of mouth.

Appendix 10

CONVERGENCE OF IMPROPER INTEGRALS

The main purpose of this appendix is to prove the comparison test, Theorem 10-6a, for the convergence of improper integrals. For the proof of this result we first establish two useful preliminary lemmas.

Lemma A10a. If every interval (a,x) contains points of the domain of ϕ, and ϕ is monotone* and bounded within some such interval, then $\lim_{x\to a^+} \phi(x)$ exists.

Similarly, if every interval (x,b) contains points of the domain of ϕ and there is at least one such interval wherein f is monotone and bounded, then $\lim_{x\to b^-} \phi(x)$ exists.

Proof. Let I be an interval (a,x_0) in which ϕ is monotone and bounded. Say that ϕ is weakly increasing in I. Since ϕ is bounded in I, it has a greatest lower bound α. Thus $\phi(x) \geq \alpha$ in I, and for every positive ϵ, there exists a point ξ in I such that

$$\phi(\xi) - \alpha < \epsilon .$$

At the same time for all x in the domain of ϕ within the interval (α,ξ) we have $\phi(x) \leq \phi(\xi)$ by the monotone property of ϕ, and $\phi(x) \geq \alpha$ because α is a lower bound; thus $0 \leq \phi(x) - \alpha \leq \phi(\xi) - \alpha < \epsilon$. We conclude that $\lim_{x\to a^+} \phi(x) = \alpha$.

The proofs of the remaining cases are left to Exercises A10, Number 1.

Lemma A10b. If f is Riemann integrable over $[\alpha,\beta]$ then $|f|$ is Riemann integrable over $[\alpha,\beta]$ (Exercises 6-4, No. 22).

---

*In A2-4, it was required that ϕ be defined on an interval for the definition of monotone function, but that requirement is not essential here.

Proof. Set

$$u(x) = \begin{cases} f(x) , & \text{if } f(x) \geq 0 ; \\ 0 , & \text{if } f(x) \leq 0 ; \end{cases}$$

$$v(x) = \begin{cases} -f(x) , & \text{if } f(x) \leq 0 ; \\ 0 , & \text{if } f(x) \geq 0 . \end{cases}$$

Thus, $|f| = u + v$. We show that u and v , hence $|f|$, are integrable. Given any ϵ there exists a partition σ of $[\alpha,\beta]$ and upper and lower sums U and L over σ such that (in the notation of Chapter 6)

$$U - L = \sum_{i=1}^{n} (M_k - m_k)(x_k - x_{k-1}) < \epsilon .$$

Let M_k^* and m_k^* denote upper and lower bounds, respectively, for $u(x)$ on $I_k = [x_{k-1}, x_k]$ and let U^* and L^* denote the corresponding upper and lower sums. We shall show that M_k^* and m_k^* can be chosen so that $M_k^* - m_k^* \leq M_k - m_k$. There are three possible cases:

(i) $f(x) \geq 0$ on I_k ; then $u(x) = f(x)$ on I_k and we take $M_k^* = M_k$, $m_k^* = m_k$.

(ii) $f(x) \leq 0$ on I_k ; then $u(x) = 0$ on I_k and we take $M_k^* = m_k^* = 0$ on I_k .

(iii) there exist points s and t in I_k such that $f(s) > 0$ and $f(t) < 0$; then $M_k \geq u(x) \geq 0 > m_k$ and we take $M_k^* = M_k$, $m_k^* = m_k$.

In each case we have $M_k^* - m_k^* \leq M_k - m_k$ so that

$$\begin{aligned} U^* - L^* &= \sum (M_k^* - m_k^*)(x_k - x_{k-1}) \\ &\leq \sum (M_k - m_k)(x_k - x_{k-1}) \\ &< \epsilon . \end{aligned}$$

There are upper and lower sums for $u(x)$ over $[a,b]$ which are closer than any assigned tolerance. It follows that u is integrable over $[a,b]$.

Since $-f$ is also integrable over $[a,b]$ it follows on replacing f by $-f$, u by $-v$ in the preceding argument that v is integrable over $[a,b]$. We conclude that $|f| = u + v$ is integrable over $[a,b]$.

THEOREM A10a. Let f be Riemann integrable over every closed interval $[\xi,\beta]$ for fixed ξ and $\beta \in (\xi,b)$. If $|f(x)| \leq g(x)$ and $\int_\xi^b g(x)dx$ converges, then $\int_\xi^b f(x)dx$ converges. Similarly, let f be Riemann integrable over every interval $[\alpha,\eta]$ for $\alpha \in (a,\eta)$. If $|f(x)| \leq g(x)$ and $\int_a^\eta g(x)dx$ converges, then $\int_a^\eta f(x)dx$ converges.

Proof. Set $K = \int_\xi^b g(x)dx$. Since $g(x)$ is nonnegative the function ψ given by

$$\psi(\beta) = \int_\xi^\beta g(x)dx$$

is weakly increasing, and since ψ has a left-sided limit at b that ψ is bounded on (ξ,b) (Exercises A10, No. 2). In particular, since ψ is bounded and monotone on (ξ,b) we conclude from the proof of Lemma A10a that

$$K = \lim_{\beta \to b^-} \psi(\beta) = \sup\{\psi(\beta) : \beta \in (\xi,b)\} ;$$

i.e., K is the least upper bound (Section A1-5) of ψ on (ξ,b), so that $\psi(\beta) = \int_\xi^\beta g(x)dx \leq K$.

Now, let u and v be the functions defined in Lemma A10b. On (ξ,b), we have

$$0 \leq u(x) \leq |f(x)| \leq g(x)$$
$$0 \leq v(x) \leq |f(x)| \leq g(x) .$$

Thus,

$$U(\beta) = \int_\xi^\beta u(x)dx \leq \psi(\beta) \leq K ,$$

$$V(\beta) = \int_\xi^\beta v(x)dx \leq \psi(\beta) \leq K .$$

Since u and v are nonnegative, we see that U and V are weakly increasing, and since U and V are bounded, Lemma A10a yields the convergence of

$$\lim_{\beta \sim b^-} U(\beta) = \int_\xi^b u(x)dx \ ,$$

$$\lim_{\beta \sim b^-} V(\beta) = \int_\xi^b v(x)dx \ .$$

But $f(x) = u(x) - v(x)$ implies the convergence of

$$\int_\xi^b f(x)dx = \int_\xi^b [u(x) - v(x)]dx \ .$$

To prove the theorem for the convergence of $\int_a^\eta f(x)dx$ make the substitution $x = -t$, $dx = -dt$, and observe that

$$\lim_{\alpha \sim a^+} \int_\alpha^\eta f(x)dx = \lim_{\beta \sim -a^-} \int_{-\eta}^\beta f(-t)dt$$

(Exercises A10, No. 3).

Theorem 10-6a is a direct consequence of Theorem A10a.

Sometimes a comparison test as defined by Theorem 10-6a is not adequate to establish the convergence of $\int_a^b f(x)dx$. Theorem A10a gives criteria for establishing absolute convergence, that is, the convergence of $\int_a^b |f(x)|dx$. However, it may happen that the integral of $f(x)$ is convergent, but not absolutely convergent.

Example 10-6b. Consider

$$I = \int_0^\infty \frac{\sin x}{x} dx$$

(the Dirichlet integral). Since $\lim_{x \sim 0} \frac{\sin x}{x} = 1$, $f : x \longrightarrow \frac{\sin x}{x}$ can be extended continuously to $x = 0$. The difficulty lies in the behavior of f for large x .

We observe that the graph of $y = \frac{\sin x}{x}$ alternates in sign as x increases and that the area under the arch of the graph over $[2n\pi, (2n + 1)\pi]$ is partly cancelled by the signed area below the x-axis and above the graph for $(2n + 1)\pi \le x \le (2n + 2)\pi$, (Figure 10-6b). It is this alternation which yields convergence.

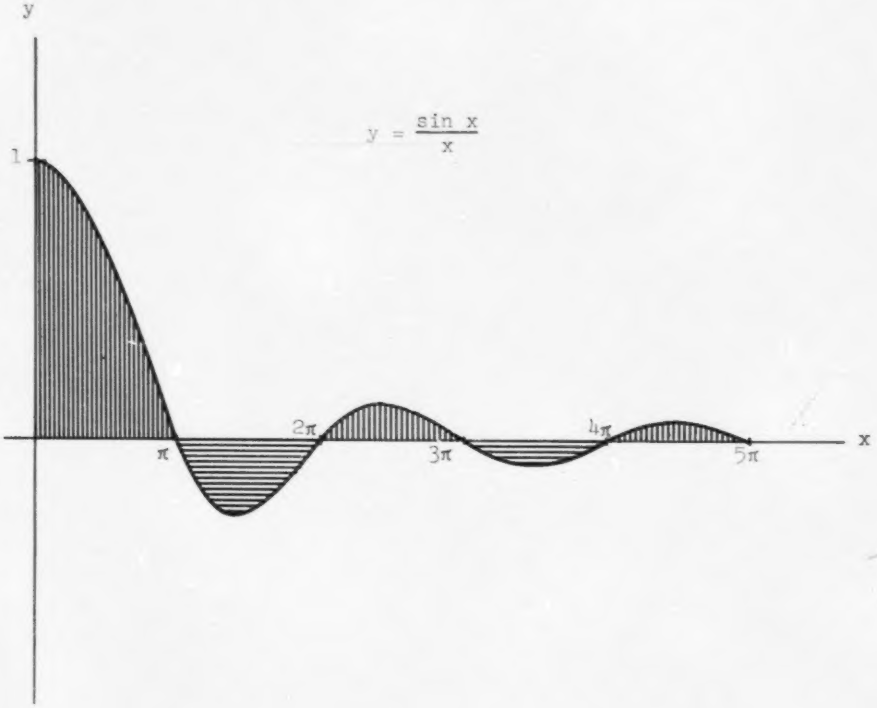

Figure A10

The convergence can be proved from this observation (Exercises A10, No. 5) but the proof can be made simple using integration by parts. To avoid the origin consider

$$J = \int_{\pi/2}^{\infty} \frac{\sin x}{x}\, dx .$$

Set $u = \frac{1}{x}$, $v = -\cos x$, $dv = \sin x \, dx$ to obtain

$$\int_{\pi/2}^{t} \frac{\sin x}{x}\,dx = -\frac{\cos t}{t} - \int_{\pi/2}^{t} \frac{\cos x}{x^2}\,dx\ .$$

Now

$$\left|\frac{\cos x}{x^2}\right| \le \frac{1}{x^2}$$

so that $\frac{\cos x}{x^2}$ is absolutely convergent over $[\frac{\pi}{2}, \infty]$, and $\lim_{t\sim\infty} -\frac{\cos t}{t} = 0$.

Consequently, $J = -\int_{\pi/2}^{\infty} \frac{\cos x}{x^2}\,dx$ converges and so does I .

Next we show that

$$\int_{0}^{\infty} \left|\frac{\sin x}{x}\right| dx$$

diverges. We have

$$\int_{0}^{n\pi} \left|\frac{\sin x}{x}\right| dx = \sum_{k=1}^{n} \int_{(k-1)\pi}^{k\pi} \left|\frac{\sin x}{x}\right| dx\ .$$

Now for x in $[(k-1)\pi + \frac{\pi}{6}, k\pi - \frac{\pi}{6}]$ we have $|\sin x| \ge \frac{1}{2}$ and $x \le k\pi$. Thus

$$\int_{(k-1)\pi}^{k\pi} \left|\frac{\sin x}{x}\right| dx \ge \int_{(k-1)\pi+\frac{\pi}{6}}^{k\pi-\frac{\pi}{6}} \left|\frac{\sin x}{x}\right| dx \ge \frac{2\pi}{3}\cdot\frac{1}{2}\cdot\frac{1}{k\pi} \ge \frac{1}{3k}$$

and hence

$$\int_{0}^{n\pi} \left|\frac{\sin x}{x}\right| dx \ge \frac{1}{3}\sum_{k=1}^{n} \frac{1}{k}\ .$$

Now we employ a trick to show that this last sum can be made arbitrarily large. From $\frac{1}{k} \ge \frac{1}{x}$ on $[k-1,k]$ we have

$$\frac{1}{k} = \int_{k-1}^{k} \frac{1}{k}\,dx \ge \int_{k-1}^{k} \frac{1}{x}\,dx\ .$$

Now summing from 1 to k we obtain,

$$\int_0^{n\pi} \left|\frac{\sin x}{x}\right| dx \geq \frac{1}{3} \sum_{k=1}^{n} \frac{1}{k}$$

$$\geq \frac{1}{3} \left[1 + \sum_{k=2}^{n} \int_{k-1}^{k} \frac{1}{x}\, dx\right]$$

$$\geq \frac{1}{3} \left[1 + \int_1^n \frac{1}{x}\, dx\right]$$

$$\geq \frac{1}{3} [1 + \log n] .$$

From the unboundedness of $\log n$ we conclude that the integral diverges.

Exercises A10

1. Complete the proof of Lemma A10-1.
2. Show if ψ is weakly increasing on (ξ, b) and has a left-sided limit at b, $K = \lim_{\beta \to b^-} \psi(\beta)$, then ψ is bounded on (ξ, b).
3. Complete the proof of Theorem A10a for $\int_a^{\eta} f(x)dx$.
4. Show that the condition in Theorem 10-6a, that f be integrable over every closed subinterval of (a,b), cannot be omitted from the convergence criterion. More precisely, show that if $|f(x)| < g(x)$ and $\int_a^b g(x)dx$ exists, then $\int_a^b |f(x)|dx$ exists but $\int_a^b f(x)dx$ need not exist.
5. By estimating the absolute difference between the areas of successive arches of the curve $y = \frac{\sin x}{x}$, i.e.,

   $$A_n = \int_{2(n-1)\pi}^{2n\pi} \frac{\sin x}{x}\, dx , \qquad (n=1,2,3,\ldots)$$

   show that

   $$I = \int_0^{\infty} \frac{\sin x}{x}\, dx$$

   converges.
6. Prove that $\int_0^1 \frac{1}{x} \sin \frac{1}{x}\, dx$ converges.

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